

THE INTEGRATED PERIODOGRAM OF A DEPENDENT EXTREMAL EVENT SEQUENCE

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ABSTRACT. We investigate the asymptotic properties of the integrated periodogram calculated from a sequence of indicator functions of dependent extremal events. An event in Euclidean space is extreme if it occurs far away from the origin. We use a regular variation condition on the underlying stationary sequence to make these notions precise. Our main result is a functional central limit theorem for the integrated periodogram of the indicator functions of dependent extremal events. The limiting process is a continuous Gaussian process whose covariance structure is in general unfamiliar, but in the iid case a Brownian bridge appears. In the general case, we propose a stationary bootstrap procedure for approximating the distribution of the limiting process. The developed theory can be used to construct classical goodness-of-fit tests such as the Grenander-Rosenblatt and Cramér-von Mises tests which are based only on the extremes in the sample. We apply the test statistics to simulated and real-life data.

1. INTRODUCTION

1.1. Regularly varying sequences. We consider a strictly stationary \mathbb{R}^d -valued sequence (X_t) for some $d \geq 1$ with a generic element X and assume that its finite-dimensional distributions are regularly varying. This means that for every $h \geq 1$, there exists a non-null Radon measure μ_h on the Borel σ -field $\overline{\mathcal{B}}_0^{dh}$ of $\overline{\mathbb{R}}_0^{dh} = \overline{\mathbb{R}}^{dh} \setminus \{0\}$, $\overline{\mathbb{R}} = \{-\infty, \infty\}$, such that

$$(1.1) \quad \frac{P(x^{-1}(X_1, \dots, X_h) \in \cdot)}{P(|X| > x)} \xrightarrow{v} \mu_h(\cdot),$$

where \xrightarrow{v} denotes vague convergence in $\overline{\mathcal{B}}_0^{dh}$; cf. Resnick [25, 26], Kallenberg [20]. The limiting measure μ_h necessarily has the property $\mu_h(t \cdot) = t^{-\alpha} \mu_h(\cdot)$, $t > 0$, for some $\alpha \geq 0$, the index of regular variation. In what follows, we assume that $\alpha > 0$. Relation (1.1) is equivalent to the sequential definition

$$(1.2) \quad n P(a_n^{-1}(X_1, \dots, X_h) \in \cdot) \xrightarrow{v} \mu_h(\cdot), \quad n \rightarrow \infty,$$

where (a_n) is chosen such that $P(|X| > a_n) \sim n^{-1}$ as $n \rightarrow \infty$. We will say that the sequence (X_t) and any of the vectors (X_1, \dots, X_h) , $h \geq 1$, are *regularly varying with index α* .

Examples of regularly varying strictly stationary sequences are linear and stochastic volatility processes with iid regularly varying noise, GARCH processes, infinite variance stable processes and max-stable processes with Fréchet marginals. These examples are discussed e.g. in Davis et al. [7, 10, 11], Mikosch and Zhao [21].

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1.2. The extremogram. Consider a μ_1 -continuity Borel set $D_0 = A \subset \overline{\mathbb{R}}_0^d$ bounded away from zero and such that $\mu_1(A) > 0$. Then the sets $D_h = A \times \overline{\mathbb{R}}^{d(h-1)} \times A$ are bounded away from zero as well and are continuity sets with respect to the corresponding limiting measures μ_{h+1} , $h \geq 1$. We conclude from (1.2) that the limits

$$(1.3) \quad \gamma_A(h) = \lim_{n \rightarrow \infty} n P(a_n^{-1} X_0 \in A, a_n^{-1} X_h \in A) = \mu_{h+1}(D_h), \quad h \geq 0,$$

exist. For $t \in \mathbb{Z}$, it is not difficult to see that

$$\begin{aligned} n \operatorname{cov}(I_{\{a_n^{-1} X_t \in A\}}, I_{\{a_n^{-1} X_{t+h} \in A\}}) &\sim n E I_{\{a_n^{-1} X_t \in A, a_n^{-1} X_{t+h} \in A\}} \\ &= n P(a_n^{-1} X_0 \in A, a_n^{-1} X_h \in A) \\ &\rightarrow \gamma_A(h), \quad n \rightarrow \infty. \end{aligned}$$

Hence γ_A constitutes the covariance function of a stationary process. We refer to γ_A as the *extremogram relative to the set A*. We will also consider the *standardized extremogram* given as the limiting sequence

$$\rho_A(h) = \lim_{n \rightarrow \infty} P(a_n^{-1} X_h \in A \mid a_n^{-1} X_0 \in A) = \frac{\mu_{h+1}(D_h)}{\mu_1(D_0)}, \quad h \geq 0.$$

The quantities $\rho_A(h)$ have an intuitive interpretation as limiting conditional probabilities. Moreover, ρ_A is the autocorrelation function of a stationary process. The quantities $\rho_A(h)$ are generalizations of the upper tail dependence coefficient of a two-dimensional vector (Y_1, Y_2) with identical marginals given as the limit $\lim_{x \rightarrow \infty} P(Y_2 > x \mid Y_1 > x)$.

The extremogram was introduced in Davis and Mikosch [7] as a measure of serial extremal dependence in a strictly stationary sequence. There and in Davis et al. [10, 11] various aspects of the estimation of the extremogram were discussed, including asymptotic theory and the use of the stationary bootstrap for the construction of confidence bands.

1.3. The sample extremogram. Natural estimators of the extremograms γ_A and ρ_A are given by their respective sample analogs

$$\tilde{\gamma}_A(h) = \frac{m}{n} \sum_{t=1}^{n-h} \tilde{I}_t \tilde{I}_{t+h} \quad \text{and} \quad \tilde{\rho}_A(h) = \frac{\tilde{\gamma}_A(h)}{\tilde{\gamma}_A(0)}, \quad h \geq 0.$$

Here $m = m_n$ is any integer sequence satisfying the conditions $m_n \rightarrow \infty$ and $m_n/n = o(1)$ and

$$I_t = I_{\{a_m^{-1} X_t \in A\}}, \quad \tilde{I}_t = I_t - p_0, \quad \text{and} \quad p_0 = E I_t = P(a_m^{-1} X \in A), \quad t \in \mathbb{Z}.$$

It is shown in Davis and Mikosch [7] that the conditions $m_n \rightarrow \infty$ and $m_n/n = o(1)$ are needed for the validity of the asymptotic properties $E \tilde{\gamma}_A(h) \rightarrow \gamma_A(h)$ and $\operatorname{var}(\tilde{\gamma}_A(h)) \rightarrow 0$ as $n \rightarrow \infty$. Moreover, under a mixing condition, the finite-dimensional distributions of $\tilde{\gamma}_A$ and $\tilde{\rho}_A$ satisfy a central limit theorem with normalization $(n/m)^{1/2}$; cf. Lemma 4 below.

1.4. Spectral density and periodogram. Since γ_A and ρ_A are the autocovariance and autocorrelation functions of a stationary process, respectively, it is possible to enter the corresponding frequency domain. If γ_A is square summable one can define the spectral densities

$$h_A(\lambda) = \sum_{h \in \mathbb{Z}} \gamma_A(h) e^{-ih\lambda} \quad \text{and} \quad f_A(\lambda) = \sum_{h \in \mathbb{Z}} \rho_A(h) e^{-ih\lambda}, \quad \lambda \in [0, \pi] = \Pi.$$

A natural estimator of the spectral density is the periodogram. Since the sample autocovariances $\tilde{\gamma}_A(h)$ are derived from the triangular array of the stationary sequences (\tilde{I}_t) , an analog of the classical

periodogram for h_A is given by

$$I_{n,A}(\lambda) = \frac{m}{n} \left| \sum_{t=1}^n \tilde{I}_t e^{-it\lambda} \right|^2 = \tilde{\gamma}_A(0) + 2 \sum_{h=1}^{n-1} \tilde{\gamma}_A(h) \cos(h\lambda), \quad \lambda \in \Pi,$$

and the periodogram for the standardized spectral density f_A is obtained as the scaled periodogram $I_{n,A}/\tilde{\gamma}_A(0)$. Mikosch and Zhao [21] showed under mixing conditions that the *extremal periodogram ordinates* $I_{n,A}(\lambda)$ share various of the classical properties of the periodogram ordinates for a stationary sequence (cf. Brockwell and Davis [3]): consistency in the mean, convergence in distribution to independent exponential random variables with expectation $h_A(\lambda_j)$ at distinct fixed frequencies $\lambda_j \in (0, \pi)$ and at distinct Fourier frequencies $\omega_n(j) = 2\pi j/n \in (0, \pi)$ provided these frequencies converge to a limit $\lambda_j \in (0, \pi)$ as $n \rightarrow \infty$. The latter property ensures that weighted versions of the periodogram $I_{n,A}$ at fixed frequencies $\lambda \in (0, \pi)$ converge in mean square to $h_A(\lambda)$.

For practical purposes, one will mostly work with the periodogram at the Fourier frequencies $\omega_n(j) \in (0, \pi)$. Then

$$I_{n,A}(\omega_n(j)) = \frac{m}{n} \left| \sum_{t=1}^n I_t e^{-it\omega_n(j)} \right|^2,$$

i.e., centering of the indicator functions I_t is not needed. However, for proving asymptotic theory it will be convenient to work with the extremal periodogram $I_{n,A}$ based on the centered quantities \tilde{I}_t , $t = 1, \dots, n$.

1.5. The integrated periodogram. The integrated periodogram of a stationary sequence has a long history in time series analysis, starting with classical work of Grenander and Rosenblatt [14], and was extensively used in the monographs Hannan [15], Priestley [24], Brockwell and Davis [3], to name a few references. Dahlhaus [4] discovered a close relationship of the integrated periodogram, considered as a process indexed by functions, and empirical process theory. Under entropy conditions, he proved uniform convergence results over suitable classes of index functions; see also the survey paper Dahlhaus and Polonik [5]. These papers gave some general theoretical background for various periodogram based techniques such as Whittle estimation of the parameters of a FARIMA process and goodness of fit tests for linear processes as mentioned in Grenander and Rosenblatt [14] and Priestley [24].

In this paper, we will consider the integrated periodogram

$$(1.4) \quad J_{n,A}(g) = \int_{\Pi} I_{n,A}(\lambda) g(\lambda) d\lambda = c_0(g) \tilde{\gamma}_A(0) + 2 \sum_{h=1}^{n-1} c_h(g) \tilde{\gamma}_A(h),$$

and its standardized version

$$J_{n,A}^{\circ}(g) = \frac{1}{\tilde{\gamma}_A(0)} \int_{\Pi} I_{n,A}(\lambda) g(\lambda) d\lambda = c_0(g) + 2 \sum_{h=1}^{n-1} c_h(g) \tilde{\rho}_A(h),$$

where g is non-negative and square integrable with respect to Lebesgue measure on Π (we write $g \in L^2_+(\Pi)$) with corresponding Fourier coefficients

$$c_h(g) = \int_{\Pi} \cos(h\lambda) g(\lambda) d\lambda, \quad h \in \mathbb{Z}.$$

We will understand $J_{n,A}(g)$ and $J_{n,A}^\circ(g)$ as natural estimators of

$$(1.5) \quad \begin{aligned} J_A(g) &= \int_{\Pi} h_A(\lambda) g(\lambda) d\lambda = c_0(g) \gamma_A(0) + 2 \sum_{h=1}^{\infty} c_h(g) \gamma_A(h), \\ J_A^\circ(g) &= \int_{\Pi} f_A(\lambda) g(\lambda) d\lambda = c_0(g) + 2 \sum_{h=1}^{\infty} c_h(g) \rho_A(h), \end{aligned}$$

respectively. The latter identities hold if $\sum_{h=0}^{\infty} \gamma_A(h) < \infty$, a condition we assume throughout this paper; see also Remark 5 below.

The main results of this paper (see Section 3) are functional central limit theorems for the integrated periodogram $J_{n,A}$ with $g = hI_{[0,\cdot]}$ for a sufficiently smooth function h on Π . The limit processes are Gaussian whose covariance structure strongly depends on the limit measures (μ_h) . The rate of convergence in these results is typically slower than \sqrt{n} . However, in the case of an iid sequence, the limiting process is a Brownian bridge and the convergence rates are much faster than in the case of a dependent sequence. These results differ from classical theory for the periodogram of a stationary sequence (X_t) (see e.g. Dahlhaus [4], Klüppelberg and Mikosch [19]), where the limiting process is completely determined by the covariance structure of (X_t) . The methods of proof combine classical techniques of weak convergence and strong mixing (e.g. Billingsley [1]) with extreme value theory for dependent sequences (e.g. Davis and Mikosch [7]). The proofs are rather technical due to the fact that the sequences of indicator functions (I_t) have triangular structure: they change in dependence on the threshold a_m .

As in classical time series analysis, the functional central limit theory for the integrated periodogram can be used to construct asymptotic goodness-of-fit tests such as the Grenander-Rosenblatt and Cramér-von Mises tests. In contrast to their classical counterparts, these tests are based only on the extremal part of the underlying sample, i.e., we test whether the extremes of the sample are in agreement with the null hypothesis about a given type of time series model. Such tests may be useful, for example, for distinguishing between a GARCH and a stochastic volatility model fitted to a return time series. The aforementioned two types of models may have similar autocorrelation structure for the data, their absolute values and squares, so their spectral properties are very similar as well, while their extremograms are rather distinct: the extremogram γ_A relative to the set $A = (1, \infty)$ decays exponentially fast for GARCH and for the simple stochastic volatility model γ_A vanishes at all positive lags; see Davis and Mikosch [7].

The paper is organized as follows. We start in Section 2 with some moment calculations and we also introduce the relevant mixing conditions and central limit theory for the sample extremogram. In Section 2.4 we provide a result about the mean square consistency of the integrated periodogram; the proof is given in Section 5. The main results (Theorems 15 and 17) are functional central limit theorems for the integrated periodogram. They are given in Section 3; the corresponding proofs are provided in Sections 6 and 7. The covariance structure of the limiting Gaussian processes in Theorem 15 is rather complicated. Therefore in Section 4 we supplement the asymptotic theory by consistency results for the stationary bootstrap applied to the integrated periodogram of extremal events in a strictly stationary sequence. The corresponding proofs are given in Section 8. In Section 4.4 we indicate how the integrated periodogram works for simulated and real-life data.

2. PRELIMINARIES

2.1. Some moment calculations. Recall the notation and conditions of Section 1. We write

$$p_0 = P(a_m^{-1} X_0 \in A) \quad \text{and} \quad p_h = P(a_m^{-1} X_0 \in A, a_m^{-1} X_h \in A), \quad h \geq 1,$$

where as above, $m_n \rightarrow \infty$ and $m_n/n = o(1)$ as $n \rightarrow \infty$. For integers $s, t, u, v \geq 0$, we set

$$\begin{aligned}\Gamma(s, t, u, v) &= E\tilde{I}_s\tilde{I}_t\tilde{I}_u\tilde{I}_v, \\ \Gamma(s, t, u) &= E\tilde{I}_s\tilde{I}_t\tilde{I}_u, \\ \Gamma(s, t) &= E\tilde{I}_s\tilde{I}_t = p_{|s-t|} - p_0^2.\end{aligned}$$

We will often have to calculate variances and covariances of the sample extremogram $\tilde{\gamma}_A$. We provide some of these formulas for further use.

Lemma 1. *Let (X_t) be a strictly stationary sequence. Then, for $1 \leq h \leq n-1$,*

$$(n/m)^2 E\tilde{\gamma}_A^2(h) = (n-h)E(\tilde{I}_0\tilde{I}_h)^2 + 2 \sum_{t=1}^{n-h-1} (n-h-t)\Gamma(0, h, t, t+h)$$

and for $1 \leq h < h+u \leq n-1$,

$$\begin{aligned}& (n/m)^2 E\tilde{\gamma}_A(h)\tilde{\gamma}_A(h+u) \\ &= (n-h-u)\Gamma(0, h, 0, h+u) \\ &+ \sum_{t=1}^{n-h-u-1} (n-h-u-t)\Gamma(0, h, t, t+h+u) \\ &+ \sum_{t=1}^{n-h-1} \min(n-h-u, n-h-t)\Gamma(0, h+u, t, t+h).\end{aligned}$$

2.2. Mixing conditions. The following two mixing conditions were introduced in Davis and Mikosch [7] for a strongly mixing \mathbb{R}^d -valued sequence (X_t) with rate function (ξ_h) .

Condition (M). There exist integer sequences $m = m_n \rightarrow \infty$ and $r_n \rightarrow \infty$ such that $m_n/n \rightarrow 0$, $r_n/m_n \rightarrow 0$ and

$$(2.1) \quad \lim_{n \rightarrow \infty} m_n \sum_{h=r_n}^{\infty} \xi_h = 0,$$

Moreover, an anti-clustering condition holds:

$$(2.2) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{h=k}^{r_n} P(|X_h| > \epsilon a_m \mid |X_0| > \epsilon a_m) = 0, \quad \epsilon > 0.$$

Condition (M1). Assume (M) and that the sequences (m_n) , (r_n) , $k_n = \lfloor n/m_n \rfloor$ from (M) also satisfy the growth conditions $k_n \xi_{r_n} \rightarrow 0$, and $m_n = o(n^{1/3})$.

Remark 2. The condition $m_n = o(n^{1/3})$ in (M1) can be replaced by $\frac{m_n r_n^3}{n} \rightarrow 0$ and $\frac{m_n^4}{n} \sum_{j=r_n}^{m_n} \xi_j \rightarrow 0$ which is often much weaker.

Condition (2.1) is easily satisfied if the mixing rate (ξ_h) is geometric, i.e., exponentially decaying to zero. Under mild conditions, the popular classes of ARMA, max-stable, GARCH and stochastic volatility processes are strongly mixing with geometric rate; cf. Davis et al. [7, 10, 11, 21] for discussions of these examples. Condition (2.2) is similar to (2.8) in Davis and Hsing [6]. It serves the purpose of establishing the convergence of a sequence of point processes to a limiting cluster point process. This condition is much weaker than the anti-clustering condition $D'(\epsilon a_n)$ of Leadbetter; cf. Section 5.3.2 in Embrechts et al. [13].

The mixing rate (ξ_h) in conditions (M) and (M1) is useful for finding bounds on the moments $\Gamma(s, t, u, v)$ introduced above. In what follows, c will denote any (possibly different) constants whose value is not of interest.

Lemma 3. *Let (X_t) be a strongly mixing sequence with mixing rate (ξ_h) . Then for integers $h, l, u \geq 1$ and for some constants $c > 0$ which do not depend on n ,*

$$(2.3) \quad |\Gamma(0, h, h+l, h+l+u)| \leq c \min(\xi_h, \xi_u),$$

$$(2.4) \quad |\Gamma(0, h, h+l, h+l+u) - (p_h - p_0^2)(p_u - p_0^2)| \leq c \xi_l,$$

$$(2.5) \quad |\Gamma(0, h, h+l)| \leq c \min(\xi_h, \xi_l),$$

$$(2.6) \quad |\Gamma(0, h)| \leq \xi_h.$$

The proof of Lemma 3 follows by a direct application of Theorem 17.2.1 in Ibragimov and Linnik [17]. Relation (2.3) combined with (2.1) will ensure that sums of $\Gamma(0, h, h+l, h+l+u)$ are asymptotically negligible if h or u exceed r_n .

2.3. Central limit theory for the sample extremogram. In this section we recall a central limit theorem for the extremogram from Davis and Mikosch [7], Section 3.

Lemma 4. *Assume that (X_t) is an \mathbb{R}^d -valued strictly stationary regularly varying sequence with index $\alpha > 0$ and that the Borel set A satisfies the conditions of Section 1.2. If the mixing conditions (M), (M1) hold and $\sum_{l=1}^{\infty} \gamma_A(l) < \infty$ then for $h \geq 0$,*

$$(2.7) \quad \tilde{\gamma}_A(h) \xrightarrow{P} \gamma_A(h),$$

$$(2.8) \quad (n/m)^{1/2} (\tilde{\gamma}_A(i) - E\tilde{\gamma}_A(i))_{i=0, \dots, h} \xrightarrow{d} (Z_i)_{i=0, \dots, h},$$

where $(Z_i)_{i=0, \dots, h}$ is Gaussian with mean zero and covariance matrix $\Sigma_h = (\sigma_{ij})_{i,j=0, \dots, h}$ given by

$$\sigma_{ij} = \gamma_A(i, j) + \sum_{l=1}^{\infty} [\gamma_A(i, l, l+j) + \gamma_A(j, l, l+i)], \quad i, j = 0, \dots, h,$$

and for $u, s, t \geq 0$,

$$\gamma_A(u, s, t) = \lim_{n \rightarrow \infty} n P(a_n^{-1} X_0 \in A, a_n^{-1} X_u \in A, a_n^{-1} X_s \in A, a_n^{-1} X_t \in A),$$

with the convention that $\gamma_A(u, t) = \gamma_A(u, u, t)$. Moreover, we have for $h \geq 1$

$$(2.9) \quad \tilde{\rho}_A(h) \xrightarrow{P} \rho_A(h),$$

$$(2.10) \quad (n/m)^{1/2} \left(\tilde{\rho}_A(i) - \frac{p_i}{p_0} \right)_{i=1, \dots, h} \xrightarrow{d} \frac{1}{\gamma_A(0)} (Z_i - \rho_A(i) Z_0)_{i=1, \dots, h}.$$

Proof. The proof of (2.7) was given in Section 3 of Davis and Mikosch [7]. There we can also find the proof of (2.8) in a more general context. Here we will calculate the covariance matrix Σ_h explicitly. The expressions for σ_{ii} , $i \geq 0$, were derived in Davis and Mikosch [7] for $i = 0$ and $i \geq 1$ in Theorem 3.1 and Lemma 5.2, respectively. We notice that $\gamma_A(i, l, l+j) \leq \gamma_A(l)$ and therefore the infinite series in σ_{ij} are finite.

For $i \neq j$, similar calculations as for Lemma 1 yield for $k \geq 1$ and $r_n/m_n \rightarrow 0$,

$$\begin{aligned}
& \frac{m}{n} \text{cov} \left(\sum_{t=1}^n \tilde{I}_t \tilde{I}_{t+i}, \sum_{s=1}^n \tilde{I}_s \tilde{I}_{s+j} \right) \\
&= m \Gamma(0, 0, i, j) + m \sum_{l=1}^n \left[(1 - l/n) [\Gamma(0, i, l, l+j) \right. \\
&\quad \left. + \Gamma(0, j, l, l+i)] - (p_i - p_0^2)(p_j - p_0^2) \right] \\
&= m \Gamma(0, 0, i, j) + m \left(\sum_{l=1}^k + \sum_{l=k+1}^{r_n} + \sum_{l=r_n+1}^n \right) \left[(1 - l/n) [\Gamma(0, i, l, l+j) \right. \\
&\quad \left. + \Gamma(0, j, l, l+i)] - (p_i - p_0^2)(p_j - p_0^2) \right] \\
&= Q_1 + Q_2 + Q_3 + Q_4.
\end{aligned}$$

By regular variation, for fixed $k \geq 1$ as $n \rightarrow \infty$,

$$Q_1 + Q_2 \rightarrow \gamma_A(i, j) + \sum_{l=1}^k [\gamma_A(i, l, l+j) + \gamma_A(j, l, l+i)],$$

and the right-hand side converges to σ_{ij} as $k \rightarrow \infty$. By (2.2), we have

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} |Q_3| = 0.$$

Using (2.4) and (2.1), we also have $|Q_4| \leq cm_n \sum_{l=r_n+1}^{\infty} \xi_l \rightarrow 0$ as $n \rightarrow \infty$. This proves (2.7) and (2.8). Relations (2.9) and (2.10) follow by a continuous mapping argument, observing that for $1 \leq i \leq h$,

$$\begin{aligned}
\left(\frac{n}{m} \right)^{1/2} (\tilde{\rho}_A(i) - p_i/p_0) &= \left(\frac{n}{m} \right)^{1/2} \frac{\tilde{\gamma}_A(i) - E\tilde{\gamma}_A(i)}{\tilde{\gamma}_A(0)} \\
&\quad - E\tilde{\gamma}_A(i) \frac{(n/m)^{1/2} (\tilde{\gamma}_A(0) - E\tilde{\gamma}_A(0))}{\tilde{\gamma}_A(0) E\tilde{\gamma}_A(0)} + o_P(1) \\
&\xrightarrow{d} \frac{1}{\gamma_A(0)} (Z_i - \rho_A(i) Z_0).
\end{aligned}$$

□

□

Remark 5. The summability condition on γ_A which we assume in the previous lemma and throughout this paper is satisfied for a large variety of regularly varying time series models; see the calculation of γ_A in Davis et al. [7, 10, 11]. For example, finite order ARMA models with regularly varying iid noise and GARCH models have exponentially decaying extremogram, and the simple stochastic volatility model with log-normal volatility process has vanishing extremogram at all positive lags. Formulas for γ_A also exist for infinite variance stable and max-stable processes with Fréchet marginals. Also for these processes the summability condition on γ_A may hold, depending on the specification of the process.

Recall that a strictly stationary process (X_t) is η -dependent for some integer $\eta \geq 0$ if $(X_t)_{t \leq 0}$ and $(X_t)_{t > \eta}$ are independent. For such a process we observe that $\sigma_{hh} = 0$ for $h > \eta$ and hence (2.8) collapses into $(n/m)^{0.5} \tilde{\gamma}_A(h) \xrightarrow{P} 0$ for $h > \eta$. In particular, for an iid sequence (X_t) , $Z_h = 0$ a.s. for $h \geq 1$, while $(n/m)^{0.5} \tilde{\gamma}_A(0) \xrightarrow{d} Z_0$ and Z_0 is $N(0, \gamma_A(0))$ distributed.

In these cases, the rate of convergence in (2.8) can be improved.

Lemma 6. Assume that (X_t) is an \mathbb{R}^d -valued η -dependent regularly varying strictly stationary sequence with index $\alpha > 0$ for some $\eta \geq 0$, and the Borel set A satisfies the conditions of Section 1.2. Additionally, assume that for $j \geq i > \eta$ and $1 \leq t \leq \eta - (j - i)$, the following limits exist:

$$\begin{aligned} & \overline{\gamma}_A(t, i, t+j) \\ &= \lim_{m \rightarrow \infty} m^2 P(a_m^{-1} X_0 \in A, a_m^{-1} X_t \in A, a_m^{-1} X_i \in A, a_m^{-1} X_{t+j} \in A). \end{aligned} \quad (2.11)$$

Then for $h \geq 1$, $n^{0.5}(\tilde{\gamma}_A(\eta + i))_{i=1, \dots, h} \xrightarrow{d} (Z_i)_{i=1, \dots, h}$, where $(Z_i)_{i=1, \dots, h}$ is Gaussian $N(0, \overline{\Sigma}_h)$ whose covariance matrix $\overline{\Sigma}_h = (\sigma_{ij})_{i,j=1, \dots, h}$ is given by

$$\begin{aligned} (2.12) \quad \sigma_{ij} &= \gamma_A(0)\gamma_A(j-i) + \sum_{t=1}^{\eta-(j-i)} [\overline{\gamma}_A(t, i, t+j) + \overline{\gamma}_A(t, j, t+i)], \\ & 1 \leq i \leq j. \end{aligned}$$

Remark 7. Condition (2.11) is an additional asymptotic independence condition. Indeed, regular variation of (X_t) only implies that the limits

$$\lim_{m \rightarrow \infty} mP(a_m^{-1} X_0 \in A, a_m^{-1} X_t \in A, a_m^{-1} X_i \in A, a_m^{-1} X_{t+j} \in A)$$

exist and are finite. Then (2.11) implies that the latter limits must be zero. In Example 9 we consider some simple cases when (2.11) is satisfied.

Remark 8. Assume $j - i > \eta$. Then, by η -dependence, $\gamma_A(j - i) = 0$ and the index set in (2.12) is empty. Hence $\sigma_{ij} = 0$ for $j - i > \eta$. In particular, if (X_t) is iid, $\sigma_{ij} = 0$ for $i \neq j$ and $\sigma_{ii} = \gamma_A^2(0)$.

Proof. We start by calculating the asymptotic covariances. Assume $j \geq i > \eta$. Then, using the independence of I_0 and $(I_j I_i, I_i I_t I_{t+j}, I_j I_t I_{t+i})$ for $t > \eta$ and of I_{t+j} and $I_0 I_t I_i$ for $t \leq \eta$ and $t \geq \eta - (j - i)$, we obtain

$$\begin{aligned} & \text{cov}(n^{0.5} \tilde{\gamma}_A(i), n^{0.5} \tilde{\gamma}_A(j)) \\ &= m^2 E \tilde{I}_0^2 E \tilde{I}_i \tilde{I}_j + m^2 \sum_{t=1}^{\eta} [E \tilde{I}_0 \tilde{I}_i \tilde{I}_t \tilde{I}_{t+j} + E \tilde{I}_0 \tilde{I}_j \tilde{I}_t \tilde{I}_{t+i}] + o(1) \\ &= \gamma_A(0)\gamma_A(j-i) + m^2 \sum_{t=1}^{\eta-(j-i)} [E \tilde{I}_0 \tilde{I}_i \tilde{I}_t \tilde{I}_{t+j} + E \tilde{I}_0 \tilde{I}_j \tilde{I}_t \tilde{I}_{t+i}] + o(1) \\ &\rightarrow \gamma_A(0)\gamma_A(j-i) + \sum_{t=1}^{\eta-(j-i)} [\overline{\gamma}_A(t, i, t+j) + \overline{\gamma}_A(t, j, t+i)], \quad n \rightarrow \infty. \end{aligned}$$

In the last step we used (2.11). This completes the calculation of $\overline{\Sigma}_h$. Furthermore, we observe that for $h \geq 1$,

$$(2.13) \quad n^{0.5}(\tilde{\gamma}_A(i))_{i=\eta+1, \dots, \eta+h} = (m/n^{0.5}) \sum_{t=1}^n (\tilde{I}_t \tilde{I}_{t+i})_{i=\eta+1, \dots, \eta+h} + o_P(1).$$

The vector sequence $(\tilde{I}_t \tilde{I}_{t+i})_{i=\eta+1, \dots, \eta+h}$, $t = 1, 2, \dots$, is strictly stationary and $(h + \eta)$ -dependent. Now an application of the central limit theorem for strongly mixing triangular arrays in Rio [27] and the Cramér-Wold device to (2.13) conclude the proof. \square \square

The following examples fulfill the conditions of Lemma 6.

Example 9. An iid regularly varying sequence (X_t) is 0-dependent, and thus (2.11) holds. Its limiting covariance matrix $\bar{\Sigma}_h$ is a diagonal matrix with entries $\gamma_A^2(0) = (\mu_1(A))^2$ on the main diagonal.

We consider the stochastic volatility model $X_t = \sigma_t V_t$ where (σ_t) is independent of (V_t) , (σ_t) is a positive η -dependent strictly stationary sequence and (V_t) is a regularly varying iid sequence with index $\alpha > 0$; see Davis and Mikosch [8]. Assume that $E\sigma^{\alpha+\varepsilon} < \infty$ for some $\varepsilon > 0$. In this case, (X_t) is η -dependent, strictly stationary and regularly varying with index α . We will show that (2.11) holds with $\bar{\gamma}_A(u, s, t) = 0$ for $0 < u < s < t$. Since A is bounded away from zero, there exists a $\delta > 0$ such that

$$\begin{aligned} & \bar{\gamma}_A(u, s, t) \\ & \leq \limsup_{m \rightarrow \infty} m^2 P(a_m^{-1} \min(|X_0|, |X_u|, |X_s|, |X_t|) > \delta) \\ & \leq \limsup_{m \rightarrow \infty} m^2 P(a_m^{-1} \max(\sigma_0, \sigma_u, \sigma_s, \sigma_t) \min(|V_0|, |V_u|, |V_s|, |V_t|) > \delta) \\ & \leq \limsup_{m \rightarrow \infty} 4m^2 P(a_m^{-1} \sigma_0 \min(|V_0|, |V_u|, |V_s|, |V_t|) > \delta) \\ & \leq \limsup_{m \rightarrow \infty} cm^2 (E\sigma^\alpha)^4 (P(|V_0| > a_m \delta))^4 = 0, \end{aligned}$$

where we used that $P(\sigma_0 | V_0| > a_m) \sim E\sigma^\alpha P(|V_0| > a_m \delta)$ by virtue of Breiman's lemma; see [2].

In the iid case, the limiting quantities Z_h , $h \geq 1$, in Lemma 4 vanish. The same observation can be made in the case of a strictly stationary sequence with asymptotic (extremal) independence in the following sense:

Condition (AI): Assume there exist sequences $m = m_n \rightarrow \infty$ and $r_n \rightarrow \infty$ such that $m = o(n)$ and $r_n = o(m)$ as $n \rightarrow \infty$ and the following conditions are satisfied for any Borel set $A \subset \mathbb{R}^d$ bounded away from zero and the axes such that $\mu_1(\partial A) = 0$:

- (1) $\lim_{n \rightarrow \infty} m^2 p_h$ exists and is finite for $h \geq 1$,
- (2) $\lim_{n \rightarrow \infty} m^2 \sup_{1 \leq i < j \leq r_n} P(a_m^{-1} X_0 \in A, a_m^{-1} X_i \in A, a_m^{-1} X_j \in A) = 0$,
- (3) $\lim_{n \rightarrow \infty} r_n m^2 \sup_{1 \leq i < j < t \leq r_n} P(a_m^{-1} X_0 \in A, a_m^{-1} X_i \in A, a_m^{-1} X_j \in A, a_m^{-1} X_t \in A) = 0$.

Example 10. We consider the stochastic volatility model from Example 9 but we drop the condition of η -dependence. Conditions (AI.2) and (AI.3) are verified in the same way as in Example 9. We also observe that for some constant $c > 0$,

$$\begin{aligned} m^2 p_h & \sim c \frac{P(a_m^{-1} X_0 \in A, a_m^{-1} X_h \in A)}{P(\min(V_1, V_2) > a_m)} \\ & = c \frac{P(a_m^{-1} \text{diag}(\sigma_0, \sigma_h)(V_1, V_2)' \in A \times A)}{P(\min(V_1, V_2) > a_m)}. \end{aligned}$$

An application of a Breiman-type result for regularly varying vectors on cones due to Janssen and Drees [18] ensures the existence and finiteness of the limits $\lim_{m \rightarrow \infty} m^2 p_h$ for $h \geq 1$. This is (AI.1).

Lemma 11. Assume that (X_t) is an \mathbb{R}^d -valued strongly mixing strictly stationary regularly varying sequence with index $\alpha > 0$ and that the Borel set A satisfies the conditions of Section 1.2. We also assume the asymptotic independence condition (AI) and the mixing condition

$$(2.14) \quad \lim_{n \rightarrow \infty} m^2 \sum_{h=r_n}^n \xi_h = 0.$$

Then

$$(2.15) \quad \begin{aligned} \tilde{\gamma}_A(h) &\xrightarrow{P} 0, \quad h \geq 1, \\ n^{0.5}(\tilde{\gamma}_A(i) - E\tilde{\gamma}_A(i))_{i=1,\dots,h} &\xrightarrow{d} (Z_i)_{i=1,\dots,h}, \end{aligned}$$

where $(Z_i)_{i=1,\dots,h}$ are independent Gaussian with mean zero and variances

$$\text{var}(Z_i) = \lim_{m \rightarrow \infty} m^2 p_i, \quad i \geq 1.$$

Proof. We will apply the central limit theorem in Rio [27] for strongly mixing triangular arrays to the left-hand side in (2.15). For this reason, we have to calculate the asymptotic covariance matrix of the left-hand vector. We observe that for fixed $j > i \geq 1$, in view of the mixing condition (2.14) as $n \rightarrow \infty$,

$$(2.16) \quad \begin{aligned} &\text{cov}(n^{0.5}\tilde{\gamma}_A(i), n^{0.5}\tilde{\gamma}_A(j)) \\ &= m^2 \text{cov}(\tilde{I}_0 \tilde{I}_j, \tilde{I}_0 \tilde{I}_i) + m^2 \sum_{t=1}^n [\text{cov}(\tilde{I}_0 \tilde{I}_i, \tilde{I}_t \tilde{I}_{t+j}) + \text{cov}(\tilde{I}_0 \tilde{I}_j, \tilde{I}_t \tilde{I}_{t+i})] + o(1) \\ &= m^2 \text{cov}(\tilde{I}_0 \tilde{I}_j, \tilde{I}_0 \tilde{I}_i) + m^2 \sum_{t=1}^{r_n} [\text{cov}(\tilde{I}_0 \tilde{I}_i, \tilde{I}_t \tilde{I}_{t+j}) + \text{cov}(\tilde{I}_0 \tilde{I}_j, \tilde{I}_t \tilde{I}_{t+i})] + o(1). \end{aligned}$$

Condition (AI) implies that $m^2 \text{cov}(\tilde{I}_0 \tilde{I}_j, \tilde{I}_0 \tilde{I}_i) \rightarrow 0$ as $m \rightarrow \infty$. The same argument also implies that the first j summands in (2.16) vanish as $n \rightarrow \infty$. Therefore it suffices to consider

$$m^2 \sum_{t=j+1}^{r_n} [\text{cov}(\tilde{I}_0 \tilde{I}_i, \tilde{I}_t \tilde{I}_{t+j}) + \text{cov}(\tilde{I}_0 \tilde{I}_j, \tilde{I}_t \tilde{I}_{t+i})].$$

In the latter sum, the indices $0, i, t, t+j$ are distinct and the same observation applies to $0, j, t, t+i$. Direct calculation with condition (AI) shows that this sum is asymptotically negligible. This implies that the covariance matrix of the limiting vector is diagonal. The calculation of the asymptotic variances is similar by observing that as $n \rightarrow \infty$,

$$\begin{aligned} &\text{var}(n^{0.5}\tilde{\gamma}_A(h)) \\ &= m^2 \text{var}(\tilde{I}_0 \tilde{I}_h) + 2m^2 \sum_{t=1}^n \text{cov}(\tilde{I}_0 \tilde{I}_i, \tilde{I}_t \tilde{I}_{t+i}) + o(1) = m^2 p_h + o(1). \end{aligned}$$

□

Remark 12. Although $\tilde{\gamma}_A(h) \xrightarrow{P} 0$, $h \geq 1$, it is in general not possible to avoid centering in (2.15). However, under (AI.1), $n^{0.5}E\tilde{\gamma}_A(h) \rightarrow 0$ if $n/m^2 = o(1)$ as $n \rightarrow \infty$, and the latter condition can even be weakened if $m^2(p_h - p_0^2) \rightarrow 0$ as $m \rightarrow \infty$.

2.4. Mean square consistency of the integrated periodogram. Recall the definitions of $J_{n,A}(g)$ and $J_A(g)$ for $g \in L_+^2(\Pi)$ from (1.4) and (1.5), respectively. The following elementary result deals with the convergence of the first and second moments of $J_{n,A}(g)$ for a given function g .

Lemma 13. Consider an \mathbb{R}^d -valued strictly stationary regularly varying sequence (X_t) with index $\alpha > 0$. Assume that the Borel set $A \subset \mathbb{R}_0^d$ satisfies the conditions of Section 1.2, $\sum_{l=1}^{\infty} \gamma_A(l) < \infty$ and (M) holds. Then the following asymptotic relations hold for $g \in L_+^2(\Pi)$.

1. $EJ_{n,A}(g) \rightarrow J_A(g)$ as $n \rightarrow \infty$.

2. If in addition, $m \log^2 n/n = O(1)$ as $n \rightarrow \infty$, and there exists a constant $c > 0$ such that

$$(2.17) \quad |c_h(g)| \leq c/h, \quad h \geq 1,$$

then $E(J_{n,A}(g) - J_A(g))^2 \rightarrow 0$ and $J_{n,A}^\circ(g) \xrightarrow{P} J_A^\circ(g)$ as $n \rightarrow \infty$.

The proof of the lemma is given in Section 5.

Remark 14. Condition (2.17) holds under mild smoothness conditions on g , e.g. if g is Lipschitz or has bounded variation on Π ; see Theorem 4.7 on p. 46 and Theorem 4.12 on p. 47 in Zygmund [30].

3. FUNCTIONAL CENTRAL LIMIT THEOREM FOR THE INTEGRATED PERIODOGRAM

Recall the definition of the spectral density h_A from Section 1.4. In this section, we assume that the weight function g is a non-negative continuous function. Abusing notation, we define the empirical spectral distribution function with weight function g by

$$(3.1) \quad J_{n,A}(x) = J_{n,A}(gI_{[0,x]}) = \int_0^x I_{n,A}(\lambda) g(\lambda) d\lambda, \quad x \in \Pi.$$

Under the conditions of Lemma 13, again abusing notation, we have

$$J_{n,A}(x) \xrightarrow{P} J_A(x) = J_A(gI_{[0,x]}) = \int_0^x h_A(\lambda) g(\lambda) d\lambda, \quad x \in \Pi.$$

In view of the monotonicity and continuity of the functions $J_{n,A}$ and J_A we also have

$$(3.2) \quad \sup_{x \in \Pi} |J_{n,A}(x) - J_A(x)| \xrightarrow{P} 0.$$

Our next goal is to complement this consistency result by a functional central limit theorem of the type $(n/m)^{0.5}(J_{n,A} - J_A) \xrightarrow{d} G$, in $\mathbb{C}(\Pi)$, the space of continuous functions on Π equipped with the uniform topology, for a suitable Gaussian limit process G .

However, this result is unlikely to hold in general, due to asymptotic bias problems. It is mentioned in Davis and Mikosch [7] in relation with the central limit theorem for the sample extremogram (see Lemma 4 above) that the pre-asymptotic centerings $E\tilde{\gamma}_A(i) = ((n-i)/n)m(p_i - p_0^2)$ can in general not be replaced by their limits $\gamma_A(i)$ due to the failure of the relation $(n/m)^{0.5}|m(p_i - p_0^2) - \gamma_A(i)| \rightarrow 0$ as $n \rightarrow \infty$. Therefore we will equip the empirical spectral distribution function $J_{n,A}$ with the pre-asymptotic centering $EJ_{n,A}$. It follows from Lemma 13 that under (M), $EJ_{n,A}(x) \rightarrow J_A(x)$ for every $x \in \Pi$, and again using monotonicity of $EJ_{n,A}$ and J_A , we have $\sup_{x \in \Pi} |EJ_{n,A}(x) - J_A(x)| \rightarrow 0$.

We observe that

$$\begin{aligned} J_{n,A}(x) &= \psi_0(x) \tilde{\gamma}_A(0) + 2 \sum_{h=1}^{n-1} \psi_h(x) \tilde{\gamma}_A(h), \\ J_{n,A}^\circ(x) &= \psi_0(x) + 2 \sum_{h=1}^{n-1} \psi_h(x) \tilde{\rho}_A(h), \end{aligned}$$

where $\psi_h(x) = \int_0^x \cos(h\lambda) g(\lambda) d\lambda$, $x \in \Pi$. We also consider a Riemann sum approximation of the coefficients $\psi_h(x)$ at the Fourier frequencies $\omega_n(i) = 2i\pi/n \in \Pi$ given by

$$\hat{\psi}_h(x) = \frac{2\pi}{n} \sum_{i=1}^{x_n} g(\omega_n(i)) \cos(h\omega_n(i)), \quad x \in \Pi,$$

where $x_n = [nx/2\pi]$. The corresponding analogs of $J_{n,A}$ and $J_{n,A}^\circ$ are then given by

$$\begin{aligned}\widehat{J}_{n,A}(x) &= \widehat{\psi}_0(x)\widetilde{\gamma}_A(0) + 2\sum_{h=1}^{n-1}\widehat{\psi}_h(x)\widetilde{\gamma}_A(h), \\ \widehat{J}_{n,A}^\circ(x) &= \widehat{\psi}_0(x) + 2\sum_{h=1}^{n-1}\widehat{\psi}_h(x)\widetilde{\rho}_A(h),\end{aligned}$$

Now we are ready to formulate the main result of this paper.

Theorem 15. *Assume that (X_t) is an \mathbb{R}^d -valued strictly stationary regularly varying sequence with index $\alpha > 0$ and the Borel set $A \subset \overline{\mathbb{R}}_0^d$ is bounded away from zero, $\mu_1(\partial A) = 0$ and $\mu_1(A) > 0$. Let g be a non-negative β -Hölder continuous function with $\beta \in (3/4, 1]$. If the conditions (M1) and $\sum_{l=1}^\infty \gamma_A(l) < \infty$ hold then in $\mathbb{C}(\Pi)$,*

$$(3.3) \quad (n/m)^{0.5}(J_{n,A} - EJ_{n,A}) \xrightarrow{d} G, \quad n \rightarrow \infty,$$

$$(3.4) \quad (n/m)^{0.5}(\widehat{J}_{n,A} - E\widehat{J}_{n,A}) \xrightarrow{d} G, \quad n \rightarrow \infty,$$

where the limit process is given by the infinite series

$$(3.5) \quad G = \psi_0 Z_0 + 2 \sum_{h=1}^\infty \psi_h Z_h,$$

which converges in distribution in $\mathbb{C}(\Pi)$, (Z_h) is a mean zero Gaussian sequence such that (Z_0, \dots, Z_h) has the covariance matrix (Σ_h) , $h \geq 0$, given in Lemma 4. Moreover, the following limit relations hold

$$(3.6) \quad (n/m)^{0.5}(J_{n,A}^\circ - EJ_{n,A}/(mp_0)) \xrightarrow{d} G^\circ, \quad n \rightarrow \infty,$$

$$(3.7) \quad (n/m)^{0.5}(\widehat{J}_{n,A}^\circ - E\widehat{J}_{n,A}/(mp_0)) \xrightarrow{d} G^\circ, \quad n \rightarrow \infty,$$

where the limit process is given by the infinite series

$$G^\circ = \frac{2}{\gamma_A(0)} \sum_{h=1}^\infty \psi_h (Z_h - \rho_A(h)Z_0).$$

The proof of this result is given in Section 6.

Remark 16. For practical purposes, the discretized version $\widehat{J}_{n,A}$ will be preferred to $J_{n,A}$ since it does not involve the calculation of integrals. Moreover, since $\sum_{t=1}^n e^{i\omega_n(j)t} = 0$ for $\omega_n(j) \in (0, \pi)$, centering of the indicators I_t with the unknown parameter p_0 in the periodogram ordinates $I_{n,A}(\omega_n(j)) = (m/n) |\sum_{t=1}^n I_t e^{i\omega_n(j)t}|^2$ is not needed.

For an η -dependent sequence (X_t) , we know that $Z_h = 0$ a.s. for $h > \eta$. Then we conclude from Theorem 15 and Lemma 4 that the limit process G collapses into $G = \psi_0 Z_0 + 2 \sum_{h=1}^\eta \psi_h Z_h$. However, taking into account Lemma 6, a more sophisticated result with a different convergence rate can be derived. The corresponding result for $J_{n,A}^\circ$ is similar and therefore omitted.

Theorem 17. *Assume that (X_t) is an \mathbb{R}^d -valued strongly mixing strictly stationary η -dependent regularly varying sequence with index $\alpha > 0$ for some $\eta \geq 0$ and the Borel set $A \subset \overline{\mathbb{R}}_0^d$ is bounded away from zero, $\mu_1(\partial A) = 0$ and $\mu_1(A) > 0$. Also assume that the limits in (2.11) exist. Let g be a*

non-negative β -Hölder continuous function with $\beta \in (3/4, 1]$. Then the relations

$$\begin{aligned}\sqrt{n}(J_{n,A} - \psi_0 \tilde{\gamma}_A(0) - 2 \sum_{h=1}^{\eta} \psi_h \tilde{\gamma}_A(h)) &\xrightarrow{d} \overline{G}, \\ \sqrt{n}(\hat{J}_{n,A} - \hat{\psi}_0 \tilde{\gamma}_A(0) - 2 \sum_{h=1}^{\eta} \hat{\psi}_h \tilde{\gamma}_A(h)) &\xrightarrow{d} \overline{G},\end{aligned}$$

hold in $\mathbb{C}(\Pi)$, where the limit process is given by the a.s. converging infinite series

$$\overline{G} = 2 \sum_{h=1}^{\infty} \psi_{\eta+h} Z_h,$$

and (Z_h) is a mean zero Gaussian sequence such that (Z_1, \dots, Z_h) has covariance matrix $\overline{\Sigma}_h$, $h \geq 1$, given in Lemma 6.

The proof is given in Section 7.

Example 18. Assume that (X_t) is an iid regularly varying sequence with index $\alpha > 0$. Then (Z_h) is an iid mean zero Gaussian sequence with $\text{var}(Z) = \gamma_A^2(0) = (\mu_1(A))^2$. If we choose the function $g \equiv 1$ we obtain

$$\psi_h(x) = \int_0^x \cos(h\lambda) d\lambda = \frac{\sin(hx)}{h}, \quad h \geq 0, \quad x \in \Pi,$$

and

$$\overline{G}(x) = 2 \sum_{h=1}^{\infty} \frac{\sin(hx)}{h} Z_h, \quad x \in \Pi.$$

We notice that \overline{G} is a series representation of a Brownian bridge; see Hida [16].

In the case of asymptotic (extremal) independence a result similar to Theorem 17 holds.

Theorem 19. Assume that (X_t) is an \mathbb{R}^d -valued strictly stationary regularly varying sequence with index $\alpha > 0$ and the Borel set $A \subset \overline{\mathbb{R}}_0^d$ is bounded away from zero and the axes, $\mu_1(\partial A) = 0$ and $\mu_1(A) > 0$. Also assume the mixing condition (2.14) and the asymptotic independence condition (AI). Let g be a non-negative β -Hölder continuous function with $\beta \in (3/4, 1]$. Then the relations

$$\begin{aligned}\sqrt{n}((J_{n,A} - EJ_{n,A}) - \psi_0(\tilde{\gamma}_A(0) - E\tilde{\gamma}_A(0))) &\xrightarrow{d} \hat{G}, \\ \sqrt{n}((\hat{J}_{n,A} - E\hat{J}_{n,A}) - \hat{\psi}_0(\tilde{\gamma}_A(0) - E\tilde{\gamma}_A(0))) &\xrightarrow{d} \hat{G},\end{aligned}$$

hold in $\mathbb{C}(\Pi)$, where the limit process is given by the a.s. converging infinite series

$$\hat{G} = 2 \sum_{h=1}^{\infty} \psi_h Z_h,$$

and (Z_h) is a sequence of independent mean zero Gaussian variables with variances $\text{var}(Z_h) = \lim_{m \rightarrow \infty} m^2 p_h$, $h \geq 1$.

The proof is based on Lemma 11 and tightness arguments which are similar to the proofs of Theorem 15 and 17. We omit further details. In view of Remark 12, centering in Theorem 19 can be avoided if $n/m^2 = o(1)$ as $n \rightarrow \infty$.

As in classical limit theory for the empirical spectral distribution (see Grenander and Rosenblatt [14], Dahlhaus [4]), an application of the continuous mapping theorem to Theorems 15 and 17 yields limit theory for functionals of the integrated periodogram. These functionals can be used for testing

the goodness of fit of the spectral density of the time series model underlying the data, under the null hypothesis that the model is correct. From Theorem 15 we get the following limit results for the corresponding test statistics.

- *Grenander-Rosenblatt test:*

$$(3.8) \quad (n/m)^{0.5} \sup_{x \in \Pi} |J_{n,A}(x) - EJ_{n,A}(x)| \xrightarrow{d} \sup_{x \in \Pi} |G(x)|.$$

- ω^2 - or Cramér-von Mises test:

$$(n/m) \int_{x \in \Pi} \left(J_{n,A}(x) - EJ_{n,A}(x) \right)^2 dx \xrightarrow{d} \int_{x \in \Pi} G^2(x) dx.$$

If (X_t) is an η -dependent sequence satisfying the conditions of Theorem 17, the corresponding limit results read as follows:

- *Grenander-Rosenblatt test:*

$$(3.9) \quad \sqrt{n} \sup_{x \in \Pi} \left| J_{n,A}(x) - \psi_0(x) \tilde{\gamma}_A(0) - 2 \sum_{h=1}^{\eta} \psi_h(x) \tilde{\gamma}_A(h) \right| \xrightarrow{d} \sup_{x \in \Pi} |\bar{G}(x)|.$$

- ω^2 -statistic or Cramér-von Mises test:

$$(3.10) \quad n \int_{x \in \Pi} \left(J_{n,A}(x) - \psi_0(x) \tilde{\gamma}_A(0) - 2 \sum_{h=1}^{\eta} \psi_h(x) \tilde{\gamma}_A(h) \right)^2 dx \xrightarrow{d} \int_{x \in \Pi} \bar{G}^2(x) dx.$$

In Figures 1 and 2 we show the estimated densities of the test statistics in (3.9) and (3.10) for $n = 2,000$ and $n = 10,000$, for different thresholds a_m and $g \equiv 1$. We compare the estimated densities with their corresponding limits. The samples are iid t -distributed with $\alpha = 3$ degrees of freedom. We mention that the density of $\sup_{x \in \Pi} |\bar{G}(x)|$ is given by $4\pi^{-2} \sum_{j=1}^{\infty} (-1)^{j+1} x \exp(-j^2 x^2 / \pi^2)$, $x > 0$; see Shorack and Wellner [28]. We use the identity in law $\int_{x \in \Pi} \bar{G}^2(x) dx \stackrel{d}{=} \sum_{j=1}^{\infty} (2/j^2) N_j^2$ for an iid standard normal sequence (N_j) (see [28]) for the simulation of the limiting density of the ω^2 -statistic.

Not surprisingly, these graphs show that one needs rather large sample sizes to make the tests reliable. The Grenander-Rosenblatt statistic shows a better overall behavior in comparison with the ω^2 -statistic. The distribution of the former statistic is close to its limit for a variety of thresholds like $p_0 = 0.1, 0.05$ and even for $p_0 = 0.03$. In contrast, the ω^2 -statistic is rather sensitive to the choice of threshold and sample size; the best overall approximation is achieved for $n = 10,000$ and $p_0 = 0.05$. For applications, one would need to focus on the quality of the approximation of high/low quantiles of the test statistics by the limiting quantiles. This task is not addressed in this paper.

4. THE BOOTSTRAPPED INTEGRATED PERIODOGRAM

With a few exceptions, the limit processes G and \bar{G} in Theorem 15 and 17 have an unfamiliar dependence structure and then it is impossible to give confidence bands for the test statistics mentioned in the previous section. One faces a similar problem when dealing with the sample extremograms whose asymptotic covariance matrix is a complicated function of the measures μ_h in (1.2). Davis et al. [10] proposed to apply the stationary bootstrap for constructing confidence bands for the sample extremogram. The stationary bootstrap can also be used for the integrated periodogram, as we will show below.

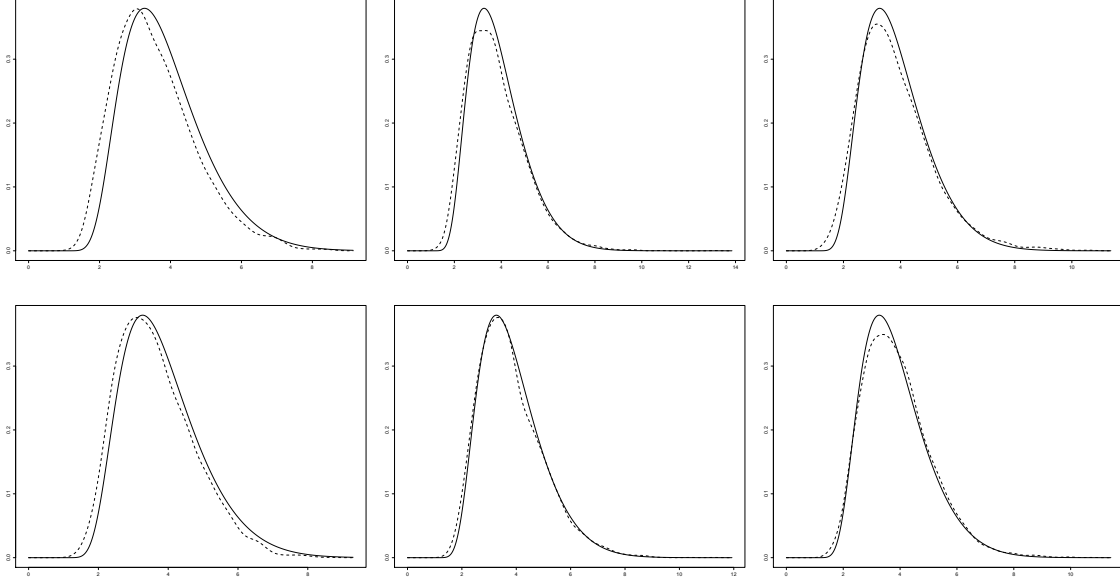


FIGURE 1. Density of the left-hand side in (3.9) with $\eta = 0$ (dotted line) and its limit $\sup_{x \in \Pi} |\bar{G}(x)|$ (solid line) for $g \equiv 1$. We choose the set $A = (1, \infty)$, different thresholds a_m with $p_0 = P(X > a_m)$ and different sample sizes n . The underlying sequence (X_t) is iid t -distributed with $\alpha = 3$ degrees of freedom. The sample sizes are chosen as $n = 2,000$ in the first row and $n = 10,000$ in the second row. The thresholds a_m are chosen such that $p_0 = 0.1$ in the first column, $p_0 = 0.05$ in the second column and $p_0 = 0.03$ in the third column.

4.1. Stationary bootstrap. The stationary bootstrap was introduced by Politis and Romano [23] as an alternative block bootstrap method. First, we describe this procedure for a strictly stationary sequence (Y_t) . Given a sample Y_1, \dots, Y_n , consider the bootstrapped sequence

$$(4.1) \quad Y_{K_1}, \dots, Y_{K_1+L_1-1}, \dots, Y_{K_N}, \dots, Y_{K_N+L_N-1}, \dots,$$

where (Y_i) , (K_i) , (L_i) are independent sequences, (K_i) is an iid sequence of random variables uniformly distributed on $\{1, \dots, n\}$, (L_i) is an iid sequence of geometrically distributed random variables with distribution $P(L_1 = i) = \theta(1 - \theta)^{i-1}$, $i = 1, 2, \dots$, for some $\theta = \theta_n \in (0, 1)$ such that $\theta_n \rightarrow 0$ as $n \rightarrow \infty$ and $N = N_n = \inf\{i \geq 1 : \sum_{j=1}^i L_j \geq n\}$. If any element Y_t in (4.1) has an index $t > n$, we replace it by $Y_{t \bmod n}$. As a matter of fact, $(Y_t)_{t \geq 1}$ constitutes a strictly stationary sequence. The stationary bootstrap sample is now chosen as the block of the first n elements in (4.1). In what follows, we write $(Y_{t^*})_{t \geq 1}$ for the bootstrap sequence (4.1), indicating that this sequence is nothing but the original Y -sequence sampled at the random indices $(K_1, \dots, K_1 + L_1 - 1, K_2, \dots, K_2 + L_2 - 1, \dots)$ with the convention that indices larger than n are taken modulo n .

In what follows, the probability measure generated by the bootstrap procedure is denoted by P^* , i.e., $P^*(\cdot) = P(\cdot | (X_t))$. The corresponding expected value, variance and covariance are denoted by E^* , var^* and cov^* .

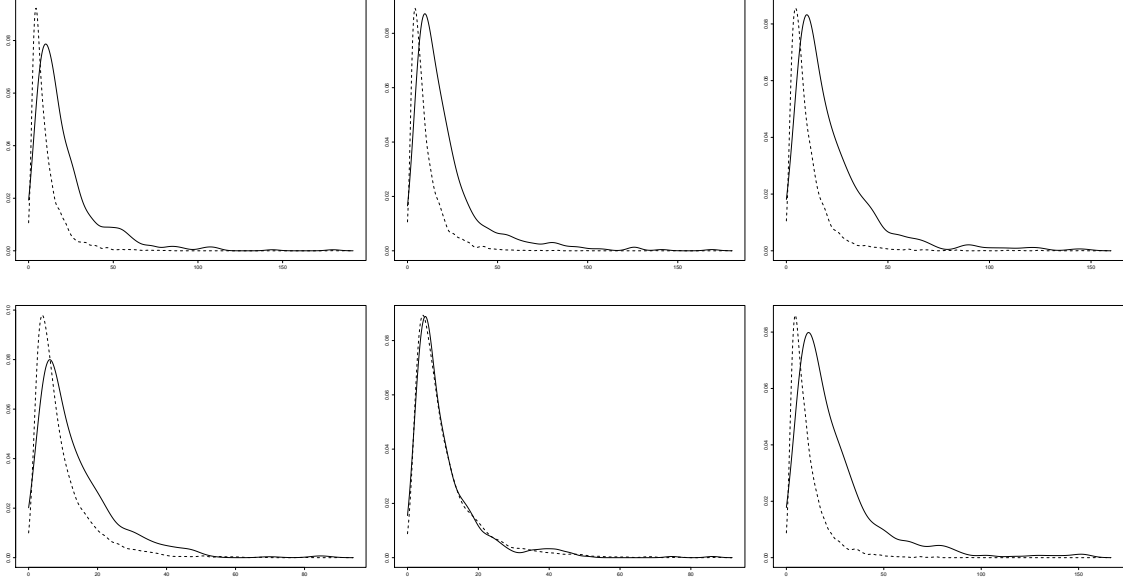


FIGURE 2. Density of the left-hand side in (3.10) with $\eta = 0$ (dotted line) and its limit $\int_{x \in \Pi} \bar{G}^2(x) dx$ (solid line) for $g \equiv 1$. We choose the same setting as in Figure 2.

4.2. The bootstrapped sample extremogram. Davis et al. [10] applied the stationary bootstrap to the sequence of lagged vectors

$$I_t(h) = (I_t^2, I_t I_{t+1}, \dots, I_t I_{t+h}), \quad t = 1, 2, \dots,$$

for fixed $h \geq 0$ and showed consistency of the bootstrapped sample extremogram. In particular, they showed the following result which we cite for further reference. A close inspection of the proof in [10] shows that the results remain true if in $I_t(h)$ we replace the quantities I_s by \tilde{I}_s , $s = t, \dots, t+h$. We denote the corresponding vector by $\tilde{I}_t(h)$. Consider the stationary bootstrap sequence $(\tilde{I}_{t^*}(h))$ and write

$$\tilde{\gamma}_A^*(i) = \frac{m}{n} \sum_{t=1}^{n-i} \tilde{I}_{t^*} \tilde{I}_{t^*+i}, \quad i = 0, \dots, h.$$

Theorem 20. Consider an \mathbb{R}^d -valued strictly stationary regularly varying sequence (X_t) with index $\alpha > 0$ and assume the following conditions:

1. The mixing condition (M1) and in addition $\sum_{h=1}^{\infty} h \xi_h < \infty$.
2. The growth conditions $\theta = \theta_n \rightarrow 0$ and $n\theta^2/m \rightarrow \infty$.
3. The set A is bounded away from zero, $\mu_1(\partial A) = 0$ and $\mu_1(A) > 0$.

Then the following bootstrap consistency results hold for $h \geq 0$:

$$E^*(\tilde{\gamma}_A^*(h)) \xrightarrow{P} \gamma_A(h) \quad \text{and} \quad \text{var}^*((n/m)^{0.5} \tilde{\gamma}_A^*(h)) \xrightarrow{P} \sigma_{hh},$$

where the covariance matrix $\Sigma_h = (\sigma_{ij})$ is given in Lemma 4. Moreover, writing d for any metric describing weak convergence in Euclidean space and $(Z_i)_{i=0, \dots, h}$ for an $N(0, \Sigma_h)$ Gaussian vector, we also have

$$d\left((n/m)^{1/2}(\tilde{\gamma}_A^*(i) - \tilde{\gamma}_A(i))_{i=0, \dots, h}, (Z_i)_{i=0, \dots, h}\right) \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

In what follows, we will write d for any metric describing weak convergence in any space of interest.

4.3. The bootstrapped integrated periodogram. Bootstrapping the sequence $(I_t(h))$ has the advantage that we preserve the neighbors I_{t^*+i} of I_{t^*} from the original sequence (I_s) . However, this method depends on the lag h and creates problems if the number of lags increases with the sample size n . In what follows, we will apply the stationary bootstrap directly to (I_t) . Then we have to re-define the bootstrap sample extremogram at any lag $h < n$. Write

$$\bar{I}_n = n^{-1} \sum_{t=1}^n I_t \quad \text{and} \quad \hat{I}_t = I_t - \bar{I}_n, \quad t \in \mathbb{Z},$$

and define the corresponding bootstrap sample extremogram

$$\hat{\gamma}_A^*(h) = \frac{m}{n} \sum_{t=1}^{n-h} \hat{I}_t^* \hat{I}_{(t+h)^*}^*, \quad h = 0, \dots, n-1,$$

and the bootstrap periodogram

$$I_{n,A}^*(\lambda) = \frac{m}{n} \left| \sum_{t=1}^n \hat{I}_t^* e^{-it\lambda} \right|^2, \quad \lambda \in \Pi.$$

Note the crucial difference: in general, $I_{t^*} I_{(t+h)^*} \neq I_{t^*} I_{t^*+h}$, but, as we will see in Lemma 25, the quantities $\hat{\gamma}_A^*(h)$ and $\hat{\gamma}_A^*(h)$ are asymptotically close for fixed $h \geq 0$.

In what follows, we focus on the bootstrap for the continuous version $J_{n,A}$ of the integrated periodogram for a given smooth weight function g ; bootstrap consistency can also be shown for the discretized version $\hat{J}_{n,A}$; we omit further details. In the definition of $J_{n,A}$ in (3.1), we simply replace (I_t) by (\hat{I}_t^*) , resulting in its bootstrap version

$$J_{n,A}^*(\lambda) = \int_0^\lambda I_{n,A}^*(x) g(x) dx = \psi_0 \hat{\gamma}_A^*(0) + 2 \sum_{h=1}^{n-1} \psi_h \hat{\gamma}_A^*(h), \quad \lambda \in \Pi.$$

Now we can formulate a bootstrap analog of Theorem 15 which shows the consistency of the stationary bootstrap procedure.

Theorem 21. *Assume the conditions of Theorem 15 and 20. Then*

$$d\left((n/m)^{1/2}(J_{n,A}^* - E^* J_{n,A}^*), G\right) \xrightarrow{P} 0, \quad n \rightarrow \infty,$$

where the Gaussian process G is defined in Theorem 15 and d is any metric which describes weak convergence in $\mathbb{C}(\Pi)$.

Remark 22. Recall that, in general, it is not possible to replace the centering $EJ_{n,A}$ of $J_{n,A}$ in the functional central limit theorem of Theorem 15 by its limit $\int_0^\lambda h_A(\lambda) g(\lambda) d\lambda$. A similar remark applies to Theorem 21. Although $\sup_{\lambda \in \Pi} |E^* J_{n,A}^*(\lambda) - J_{n,A}(\lambda)| \xrightarrow{P} 0$, under the conditions of Theorem 21, it is in general not possible to replace the centering $E^* J_{n,A}^*$ by $J_{n,A}$; see Lemma 28. Thus, Theorem 21 does not yield bootstrap consistency in a textbook sense but it rather provides a simulation technique for the limit process G . In turn, the simulation of this process makes it possible to provide confidence bands for the goodness of fit test statistics considered above. We will apply this simulation procedure in Section 4.4.

4.4. A simulation study. We focus on the Grenander-Rosenblatt statistic (GRS) on the left-hand side of (3.8) for different time series models, distinct thresholds and sample sizes. Under the null hypothesis of a particular time series model, one can simulate the quantiles of the GRS from the theoretical model. In this study we also follow a different approach. First, we determine the expected value function $EJ_{n,A}$ and the threshold a_m such that $p_0 = P(X > a_m) = 1/m$ by simulation from the theoretical model and then we use the stationary bootstrap to calculate the asymptotic quantiles of the GRS. This distribution is obtained by repeated simulation of $(n/m)^{0.5} \sup_{x \in \Pi} |J_{n,A}^*(x) - E^* J_{n,A}^*(x)|$; Theorem 21 provides a justification for this approach.¹ In the cases when the expected value function $EJ_{n,A}$ can be replaced by its limit, i.e., when the bias of $J_{n,A}$ is negligible, this approach has the advantage that the test is non-parametric. An example are models satisfying the asymptotic independence condition (AI) and $n/m^2 \rightarrow 0$ as $n \rightarrow \infty$; see Theorem 19 and the remark following it. Of course, for an iid sequence or η -dependent sequence one can also use the quantiles of the limit distribution of the GRS which are known or can be simulated; see (3.9) and (3.10).

In what follows, we apply the Grenander-Rosenblatt test (GRT) to various univariate (real-life or simulated) time series $X_t, t = 1, \dots, n$ for different sample sizes n and thresholds a_m . We always choose $A = (1, \infty)$ and $g \equiv 1$. Whenever we apply the stationary bootstrap we choose the geometric parameter $\theta = 1/50$. Density plots and simulated quantiles are derived from 4,000 independent repetitions, also in the bootstrap case.

In Figure 3 we illustrate how the stationary bootstrap works for different thresholds a_m and sample size $n = 2,000$. We show the density of the normalized GRS on the left-hand side of (3.8) and its bootstrap approximation. We choose regularly varying ARMA(1, 1) and GARCH(1, 1) models. The densities of the GRS and its bootstrap approximation are close to each other. We take this fact as justification for using the bootstrap quantiles of the GRS in the test. While the densities in the ARMA case do not seem too sensitive to the choice of the high threshold a_m , the shape of the densities change for the GARCH model when switching from $p_0 = 0.10$ to $p_0 = 0.05$, while they look similar for $p_0 = 0.05$ and $p_0 = 0.01$.

In Figure 4 we show sample paths of the normalized and centered integrated periodogram $(n/m)^{0.5} |J_{n,A} - EJ_{n,A}|$ with $p_0 = 0.05$ for samples of size $n = 2,000$ from ARMA(1,1) and GARCH(1,1) models together with 95%-quantiles of the GRS both under the correct and under an incorrect null hypothesis. Due to the need of centering with $EJ_{n,A}$ these sample paths are affected both by the sample and the model. Indeed, if the model is chosen incorrectly we will typically subtract the incorrect centering and calculate an incorrect threshold a_m . When using both the bootstrap-based or true 95%-quantiles of the GRS, the model is not rejected if the sample is in agreement with the null hypothesis. However, if the sample comes from a model whose parameters slightly deviate from the parameters of the null hypothesis the incorrect expected value $EJ_{n,A}$ and wrong threshold a_m change the sample path of the integrated periodogram in such a way that the bootstrap-based GRT rejects the null hypothesis while it does not reject the null if one uses the quantiles based on the null hypothesis. It is advantageous to show both 95%-quantiles: they deviate rather significantly, indicating that we chose an incorrect null model.

In Figure 5 we consider a stochastic volatility model $X_t = \sigma_t Z_t$, where (σ_t) is a log-normal stationary process independent of the iid t -distributed sequence (Z_t) . The α degrees of freedom of the t -distribution coincide with the index α of regular variation of (X_t) . The extremogram of this sequence vanishes at all positive lags. This fact is in agreement with the extremogram of an iid sequence but is in contrast to a GARCH(1, 1) process. Choosing $p_0 = 0.05$, we apply the GRS to a stochastic volatility sample of size $n = 2,000$ under the incorrect null hypothesis of a GARCH(1, 1)

¹ Throughout, to exploit the power of the Fast Fourier Transform, we use the Riemann sum approximations to the integrated periodograms. We do not indicate this fact in the notation.

model with tail index close to the chosen α . The test clearly rejects the null hypothesis. We also run a GRT for the stochastic volatility sample under the null hypothesis of an iid t -distributed sequence with α degrees of freedom. We use the approximation of the distribution of the GRS by the distribution of the supremum of a Brownian bridge; see Example 18. Also in this case, the null is clearly rejected.

In Figure 6 we deal with a time series (X_t) of 1,560 1-minute log-returns of Goldman Sachs stock from the period November 7-10, 2011. It has estimated tail index $\alpha \approx 3$. Using standard software, we fitted a GARCH(1,1) model such that $\sigma_t^2 = 0.019 + 0.1X_{t-1}^2 + 0.87\sigma_{t-1}^2$. Hill and QQ plots of the residuals of this model indicate that the noise is well fitted by a t -distribution with (approximately) 4 degrees of freedom. The theoretical index of regular variation of this GARCH(1,1) model is $\alpha = 3.13$; see Table 2 in [9]. We test the null hypothesis of a GARCH(1,1) model with the aforementioned parameters. This hypothesis is rejected. On the other hand, the GRT passes under the hypothesis of an iid sequence, where we choose a_m as the 95% empirical quantile. This means that the extremes of this data set are more in agreement with an iid than with a GARCH structure. This is perhaps not surprising in view of a high frequency data return series while GARCH seems more suitable for fitting low frequency returns.

A GARCH(1,1) model is often considered to give a good fit to daily log-returns of stock prices and foreign exchange (FX) rates. For example, such a judgement may be based on tests for zero autocorrelation of the residuals, their absolute values and squares. We did not find evidence of GARCH behavior in the extremes of three 5-year time series of daily Euro-USD FX rate log-returns: from 2002 to 2006 (before the financial crisis), from 2006 to 2010 (including the financial crisis), from 2009 to 2013 (after the financial crisis); see Figure 7. We choose different thresholds a_m . When $p_0 = 0.05$ the null hypothesis of an iid sequence is accepted for 2002-2006 and 2009-2013, but not for 2006-2010. The null hypothesis of a fitted GARCH process with $\sigma_t^2 = 2.37 \times 10^{-7} + 0.1X_{t-1}^2 + 0.8\sigma_{t-1}^2$ and iid t -distributed noise with 4 degrees of freedom is also rejected by the GRT for 2006-2010. For this latter period, the stationarity assumption may be doubted. We repeat the GRTs for $p_0 = 0.02$ in the periods 2002-2006 and 2009-2013. In the latter case the iid null hypothesis is still not rejected while it is rejected in the former case. The abrupt change of the behavior of the GRT may be due to the sample size (roughly 1,280 for each time series). For $p_0 = 0.02$ one would use only 2% of the data for the calculation of the GRT.

Our simulation study points at some of the problems one has to face when using goodness of fit tests based on the extremes of a time series. A major problem is the choice of the threshold a_m . A data driven choice would be preferable but we do not have a theoretical answer to the problem. We propose to use graphical methods to compare the shapes of the extremogram and the integrated periodogram for different thresholds and to choose a sufficiently high threshold where the shapes stabilize. A message from the simulations is that the sample size n should not be too small. For example, the GRTs in Figure 7 with $n \approx 1,280$ give rather distinct answers when switching from $p_0 = 0.05$ to $p_0 = 0.02$. The sample extremogram and the integrated periodogram render meaningless for too high thresholds because most indicator functions of extreme events will be zero. The simulation study indicates that it is useful to exploit the true quantiles of the GRS (obtained by simulation from a model under the null hypothesis) as well as corresponding bootstrap-based quantile of the GRS. In particular, when the null hypothesis is incorrect the two 95% quantiles (say) will typically differ, pointing at the incorrect null hypothesis. We do not address the problem

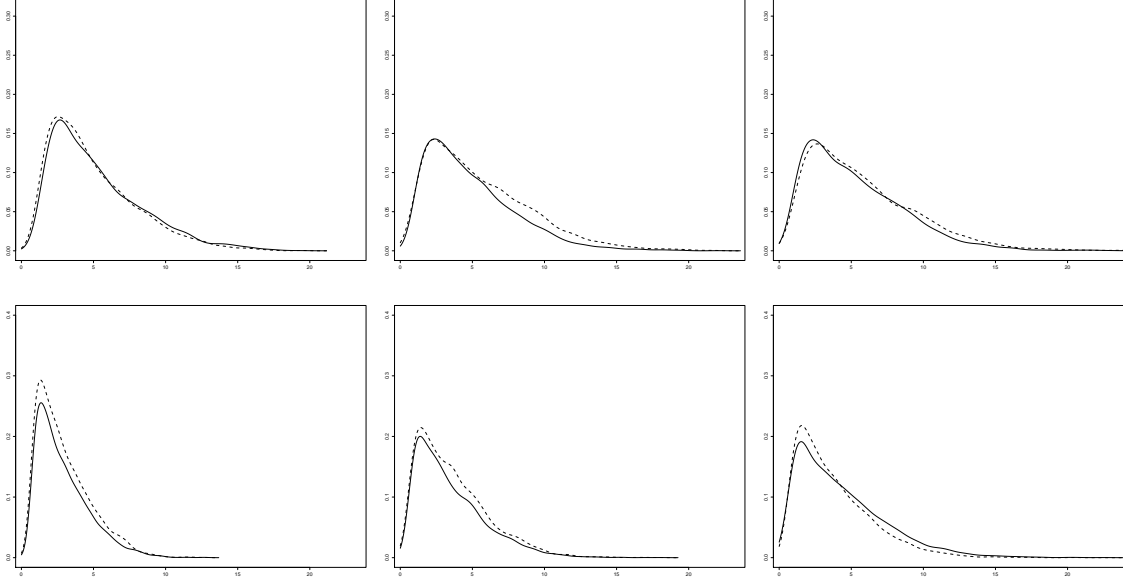


FIGURE 3. Density of the normalized GRS (solid line) and its bootstrap approximation. The sample size is $n = 2,000$. and the thresholds a_m are chosen such that $p_0 = P(X > a_m) = 0.10, 0.05, 0.03$ corresponding to the first, second and third column. *Top*: The sample is drawn from the ARMA(1,1) process $X_t = 0.8X_{t-1} + 0.1Z_{t-1} + Z_t$, where (Z_t) is iid t -distributed with $\alpha = 3$ degrees of freedom. *Bottom*: The sample is drawn from the GARCH(1,1) process $X_t = \sigma_t Z_t$, where $\sigma_t^2 = 0.1 + 0.1X_{t-1}^2 + 0.84\sigma_{t-1}^2$ and (Z_t) is iid t -distributed with 4 degrees of freedom. In this case, the index of regular variation for (X_t) is $\alpha = 3.49$; see Table 2 in Davis and Mikosch [9].

of goodness of fit tests in the case when the null hypothesis depends on estimated parameters; the asymptotic theory does not change under mild conditions on the convergence rates of the estimators.

5. PROOF OF LEMMA 13

Part 1. Recall the series representations of $J_{n,A}(g)$ and $J_A(g)$ from (1.4) and (1.5), respectively. Then for every fixed $k \geq 1$, large n ,

$$\begin{aligned}
 J_{n,A}(g) - J_A(g) &= \left(c_0(g)[\tilde{\gamma}_A(0) - \gamma_A(0)] + 2 \sum_{h=1}^k c_h(g) [\tilde{\gamma}_A(h) - \gamma_A(h)] \right) \\
 &\quad + 2 \sum_{h=k+1}^{n-1} c_h(g) [\tilde{\gamma}_A(h) - \gamma_A(h)] - 2 \sum_{h=n}^{\infty} c_h(g) \gamma_A(h) \\
 &= I_1(k) + I_2(k) - I_3.
 \end{aligned}$$

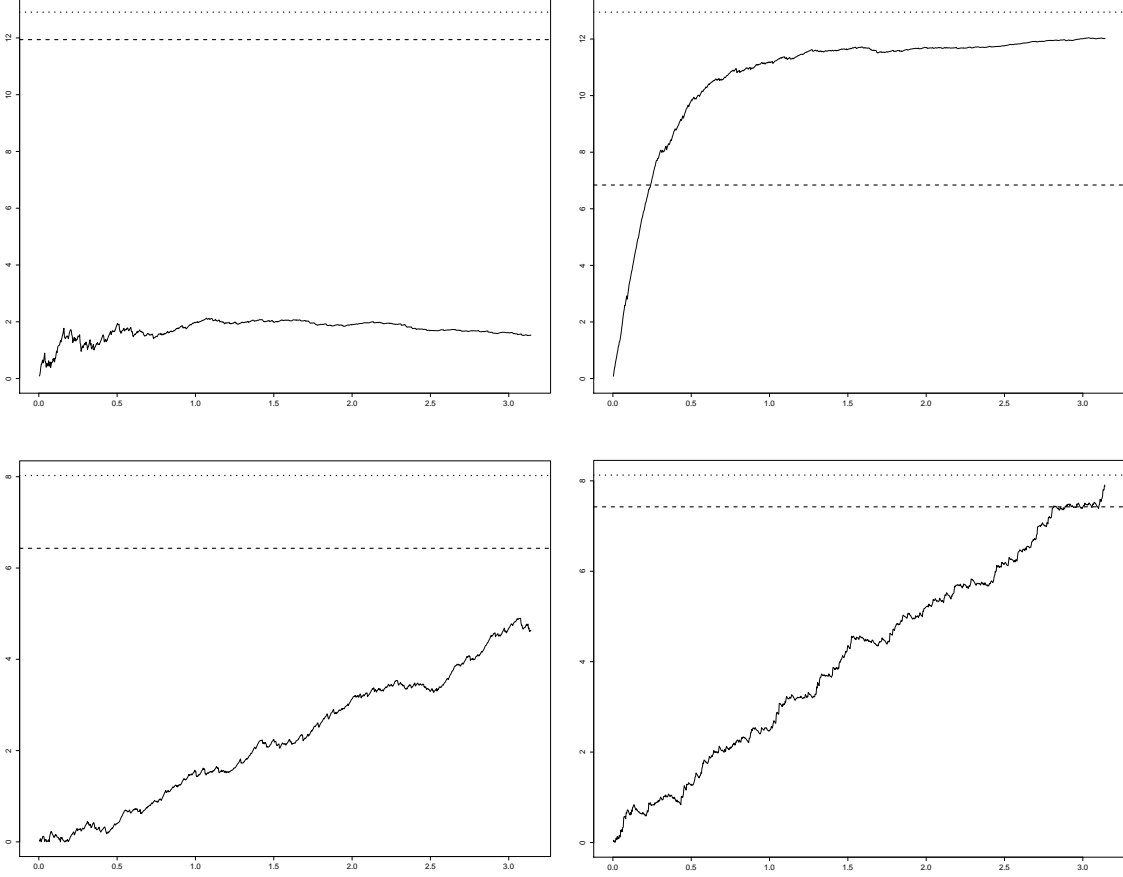


FIGURE 4. Paths of the integrated periodogram $(n/m)^{0.5}|J_{n,A} - EJ_{n,A}|$ with $p_0 = 0.05$ for samples of size $n = 2,000$. **Top:** We work under the null hypothesis of the ARMA(1,1) model $X_t = 0.8X_{t-1} + 0.3Z_{t-1} + Z_t$, where (Z_t) is iid t -distributed with $\alpha = 3$ degrees of freedom. *Left:* The sample is drawn from the null model. The lower and upper dotted lines $y = 11.9$ and $y = 12.9$ correspond to the bootstrap-based and true 95%-quantiles of the GRS, respectively. The null hypothesis would be accepted. *Right:* The sample is drawn from the ARMA(1,1) process $X_t = 0.8X_{t-1} + 0.1Z_{t-1} + Z_t$ with the same distribution for (Z_t) . The lower dotted line $y = 6.84$ is the bootstrap-based 95%-quantile of the GRS. Based on it, the test would reject the null. However, it would accept the null if one chose the 95%-quantile of the null model. **Bottom:** We work under the null hypothesis of the GARCH(1,1) process $X_t = \sigma_t Z_t$, where $\sigma_t^2 = 10^{-7} + 0.1X_{t-1}^2 + 0.81\sigma_{t-1}^2$ and (Z_t) is iid t -distributed with 4 degrees of freedom. *Left:* The sample is chosen from the null model. The lower and upper dotted lines $y = 6.4$ and $y = 8$ correspond to the bootstrap-based and true 95%-quantiles of the GRS, respectively. The null would be accepted for both quantiles. *Right:* The sample is drawn from a GARCH(1,1) process with $\sigma_t^2 = 10^{-7} + 0.1X_{t-1}^2 + 0.84\sigma_{t-1}^2$ and the same distribution of (Z_t) . The lower dotted line $y = 7.4$ is the bootstrap-based 95%-quantile of the GRS. The null would be rejected in this case while it would be accepted if one used the 95%-quantile $y = 8$ based on the null hypothesis.

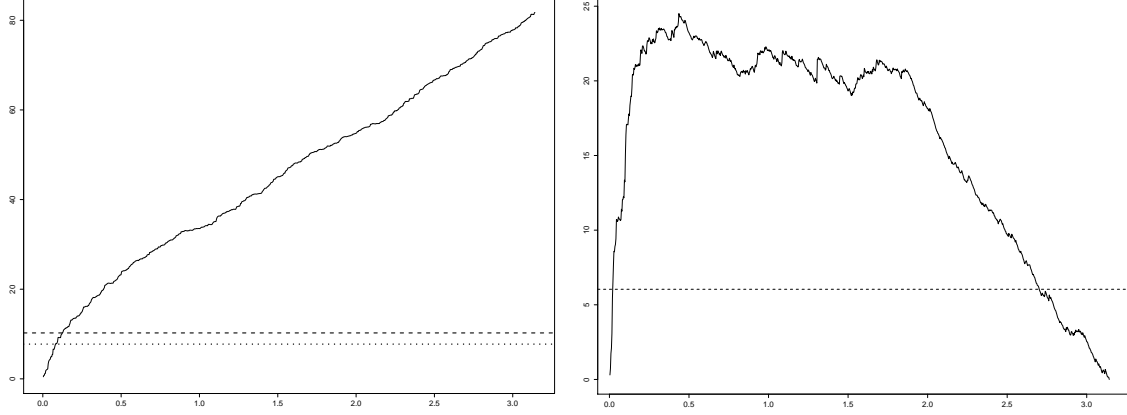


FIGURE 5. The sample of size $n = 2,000$ is drawn from a stochastic volatility process $X_t = \sigma_t Z_t$ with log-volatility $\log \sigma_t = 0.9 \log \sigma_{t-1} + \epsilon_t$ for an iid standard normal sequence (ϵ_t) , Z_t is t -distributed with 3.6 degrees of freedom. *Left:* Sample path of $(n/m)^{0.5}|J_{n,A} - EJ_{n,A}|$ with $p_0 = 0.05$. The lower and upper dotted lines $y = 7.8$ and $y = 10.2$ correspond to the true and bootstrap-based 95%-quantiles of the GRS under the null hypothesis of a GARCH(1,1) process $\tilde{X}_t = \tilde{\sigma}_t \tilde{Z}_t$ with $\tilde{\sigma}_t^2 = 6.23 \times 10^{-3} + 0.1\tilde{X}_{t-1}^2 + 0.8\tilde{\sigma}_{t-1}^2$ and iid t -distributed (\tilde{Z}_t) with 4 degrees of freedom. This process has tail index 3.68; see Table 1 in [9]. The test clearly rejects the null hypothesis. *Right:* Sample path of the integrated periodogram absolute value $n^{0.5}|J_{n,A} - \psi_0 \tilde{\gamma}_A(0)|$. The dotted line is the 95%-quantile of the distribution of the supremum of the absolute values of a Brownian bridge. The test clearly rejects the null hypothesis that (X_t) is iid.

Then $I_3 \rightarrow 0$ as $n \rightarrow \infty$ since $(\gamma_A(h))$ is summable and $EI_1(k)$ converges to zero as $n \rightarrow \infty$ due to regular variation, for every k . In view of (2.1) in (M),

$$\begin{aligned} \left| E \sum_{h=r_n+1}^{n-1} \tilde{\gamma}_A(h) c_h(g) \right| &= \left| \frac{m}{n} \sum_{h=r_n+1}^{n-1} (n-h) c_h(g) (p_h - p_0^2) \right| \\ &\leq c m \sum_{h=r_n+1}^{\infty} \xi_h \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

and (2.2) in (M) implies

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| E \sum_{h=k+1}^{r_n} \tilde{\gamma}_A(h) c_h(g) \right| \leq c \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} m \sum_{h=k+1}^{r_n} p_h = 0.$$

Since $\lim_{k \rightarrow \infty} \sum_{h=k+1}^{\infty} \gamma_A(h) = 0$, we have $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} |EI_2(k)| = 0$. This proves Part 1.

Part 2. It follows from Theorem 3.1 in Davis and Mikosch [7] that $\tilde{\gamma}_A(h) \xrightarrow{L^2} \gamma_A(h)$, $h \geq 1$. Hence $I_1(k) \xrightarrow{L^2} 0$ as $n \rightarrow \infty$ for fixed $k \geq 1$. It remains to show that $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \text{var}(I_2(k)) = 0$.

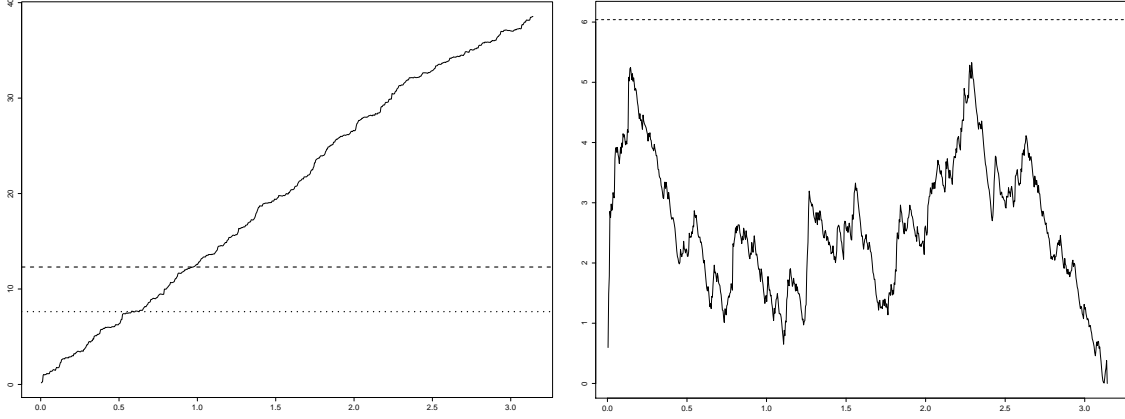


FIGURE 6. GRTs for 1,560 Goldman Sachs 1-minute log-returns. *Left:* The integrated periodogram $(n/m)^{0.5}|J_{n,A} - EJ_{n,A}|$ with $p_0 = 0.05$ under the null hypothesis that the data are generated by the GARCH(1,1) model $\sigma_t^2 = 0.019 + 0.1X_{t-1}^2 + 0.87\sigma_{t-1}^2$ with iid t -distributed noise with 4 degrees of freedom. The lower and upper dotted lines $y = 7.6$ and $y = 12.3$ represent the true and bootstrap-based 95%-quantiles of the GRS under the null hypothesis. The hypothesis of GARCH(1,1) is clearly rejected. *Right:* The integrated periodogram $n^{0.5}|J_{n,A} - \psi_0 \tilde{\gamma}_A(0)|$ with $p_0 = 0.05$ under the null hypothesis of an iid sequence. The dotted line represents the asymptotic 95%-quantile based on the approximation of the GRS by the supremum of the absolute values of a Brownian bridge. The null hypothesis is not rejected.

We have

$$I_2(k) = 2 \left(\sum_{h=k+1}^{r_n} + \sum_{h=r_n+1}^{n-1} \right) c_h(g) [\tilde{\gamma}_A(h) - \gamma_A(h)] = 2I_{21}(k) + 2I_{22}.$$

In view of Lemma 1 we get the bound

$$\begin{aligned} \text{var}(I_{21}(k)) &\leq \frac{m^2}{n} \sum_{h=k+1}^{r_n} \sum_{l=0}^{r_n-h} |c_h(g)c_{h+l}(g)| \times \\ &\quad \left(|\Gamma(0, h, 0, h+l)| + \sum_{t=1}^{n-h-l} |\Gamma(0, h, t, t+h+l)| + \sum_{t=1}^{n-h} |\Gamma(0, h+l, t, t+h)| \right) \\ &= Q_1 + Q_2 + Q_3. \end{aligned}$$

Since $|c_h(g)| \leq c/h$ (see (2.17)),

$$\begin{aligned} |Q_1| &\leq c \frac{m^2}{n} \sum_{h=k+1}^{r_n} |c_h(g)| \sum_{s=h}^{r_n} |c_s(g)| p_s = c \frac{m^2}{n} \sum_{s=k+1}^{r_n} |c_s(g)| p_s \sum_{h=k+1}^s |c_h(g)| \\ &\leq c \frac{m^2}{n} \sum_{s=k+1}^{r_n} p_s s^{-1} \log s, \end{aligned}$$

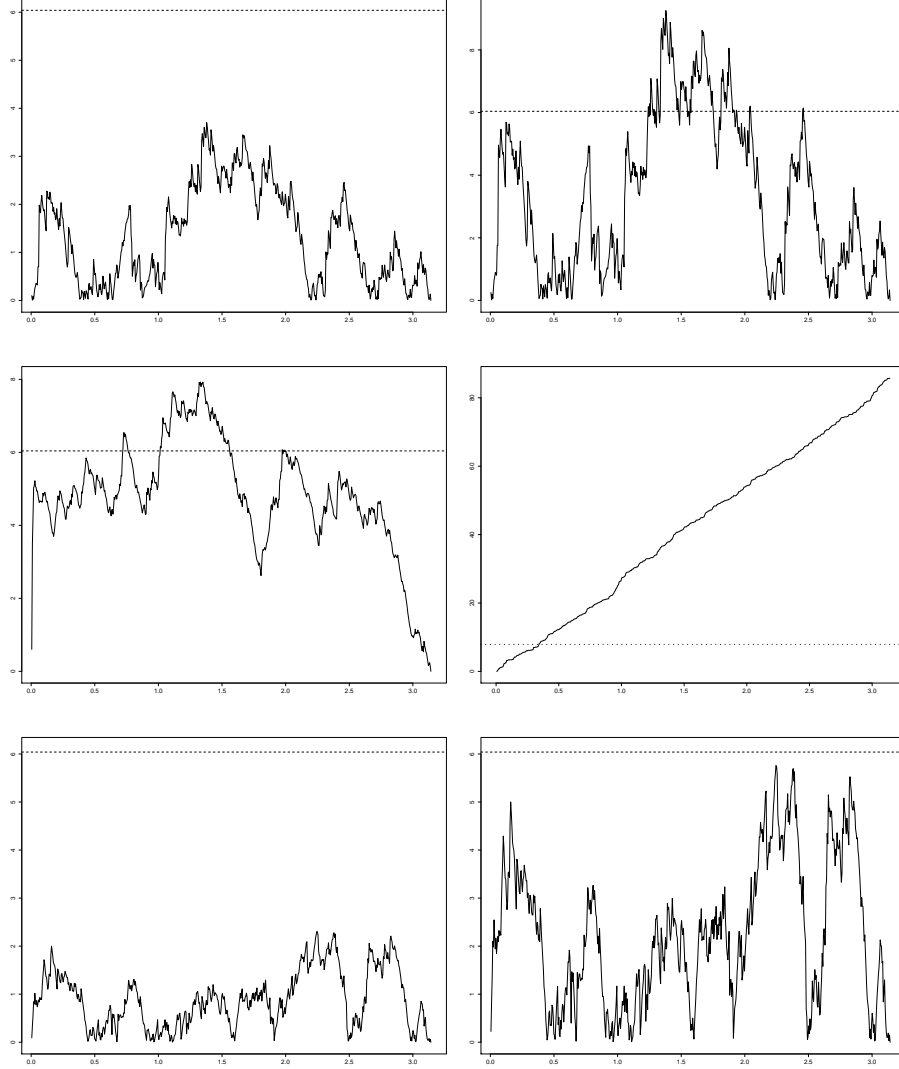


FIGURE 7. GRTs for daily Euro-USD FX rate log-returns 2002-2006(top, $n = 1,280$), 2006-2010 (middle, $n = 1,279$), 2009-2013 (bottom, $n = 1,281$). The graphs show the integrated periodograms $n^{0.5}|J_{n,A} - \psi_0 \tilde{\gamma}_A(0)|$ under the null hypothesis of an iid sequence and $(n/m)^{0.5}|J_{n,A} - EJ_{n,A}|$ in the case of a fitted GARCH model. Under the iid hypothesis, the dotted lines represent the 95%-quantile obtained from the limiting supremum of the absolute values of a Brownian bridge. Under the GARCH hypothesis, the dotted line represents the bootstrap-based 95%-quantile of the GRS. **Top:** FX rate log returns 2002-2006 ($n = 1,280$). We test under the iid null hypothesis. For $p_0 = 0.05$ (left), the null is not rejected. This is in contrast to the case $p_0 = 0.02$ (right) which leads to a clear rejection. The qualitative difference may be due to the relatively small sample size which renders the test statistics meaningless. **Middle:** FX rate log returns 2006-2010 ($n = 1,279$). Left. The iid null hypothesis with $p_0 = 0.05$ is rejected. Right. A GARCH(1,1) model with $\sigma_t^2 = 2.37 \times 10^{-7} + 0.1X_{t-1}^2 + 0.8\sigma_{t-1}^2$ and iid t -distributed noise with 4 degrees of freedom is fitted to the data. The null hypothesis of this GARCH is clearly rejected. **Bottom:** FX rate log returns 2009-2013 ($n = 1,281$). The iid null hypothesis with $p_0 = 0.05$ (left) and $p_0 = 0.02$ (right) is not rejected.

and the right-hand side converges to 0 by first letting $n \rightarrow \infty$ and then $k \rightarrow \infty$, using (2.2). Since the structures of Q_2 and Q_3 are similar we restrict ourselves to showing $Q_2 \rightarrow 0$ as $n \rightarrow \infty, k \rightarrow \infty$. We observe that

$$\begin{aligned} |Q_2| &\leq c \frac{m^2}{n} \sum_{h=k+1}^{r_n} \sum_{s=h}^{r_n} \frac{1}{hs} \left(\sum_{t=1}^{2r_n} + \sum_{t=2r_n+1}^n \right) |\Gamma(0, h, t, t+s)| \\ &\leq c \frac{m \log^2 r_n}{n} m \sum_{h=k+1}^{3r_n} p_h + c \frac{m \log^2 r_n}{n} m \sum_{h=r_n+1}^n \xi_h + cn^{-1} \left(m \sum_{h=k+1}^{r_n} p_h/h \right)^2. \end{aligned}$$

In the last step, we used (2.4). The right-hand side vanishes as $n \rightarrow \infty$ and $k \rightarrow \infty$. Finally, we conclude that $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \text{var}(I_{21}(k)) = 0$.

Now we turn to bounding $\text{var}(I_{22})$. In view of Lemma 1 we have

$$\begin{aligned} \text{var}(I_{22}) &\leq \frac{m^2}{n} \sum_{h=r_n+1}^{n-1} \sum_{s=h}^{n-1} |c_h(g)c_s(g)| \left(|\Gamma(0, h, 0, s)| + \sum_{t=1}^{n-s} |\Gamma(0, h, t, t+s)| \right. \\ &\quad \left. + \sum_{t=1}^{n-h} |\Gamma(0, s, t, t+h)| \right) = Q_4 + Q_5 + Q_6. \end{aligned}$$

We have by (2.17),

$$\begin{aligned} Q_4 &\leq c \frac{m^2}{n} \sum_{h=r_n+1}^{n-1} \sum_{s=h}^{n-1} |c_h(g)c_s(g)| |E\tilde{I}_0\tilde{I}_s| \\ &\leq c \frac{m^2}{n} \sum_{h=r_n+1}^{n-1} h^{-2} \sum_{s=h}^{n-1} [(p_s - p_0^2) + p_0^2] \\ &\leq c \left[\frac{m}{nr_n} m \sum_{h=r_n+1}^{\infty} \xi_h + \frac{(p_0 m)^2}{r_n} \right] = o(1), \quad n \rightarrow \infty. \end{aligned}$$

The terms Q_5 and Q_6 can be treated in a similar way; we focus on Q_5 . By (2.17),

$$\begin{aligned} Q_5 &\leq c \frac{m^2}{n} \sum_{h=r_n+1}^{n-1} \sum_{s=h}^{h+r_n} (hs)^{-1} \sum_{t=1}^{r_n} |\Gamma(0, h, t, t+s)| \\ &\quad + c \frac{m^2}{n} \sum_{h=r_n+1}^{n-1} \sum_{s=h+1}^{n-1} \sum_{t=r_n+1}^{n-s} (hs)^{-1} |\Gamma(0, h, t, t+s)| \\ &\quad + c \frac{m^2}{n} \sum_{h=r_n+1}^{n-1} \sum_{s=h+r_n+1}^{n-1} \sum_{t=1}^{r_n} (hs)^{-1} |\Gamma(0, h, t, t+s)| \\ &= Q_{51} + Q_{52} + Q_{53}, \end{aligned}$$

and

$$\begin{aligned} Q_{51} &\leq c \frac{m^2}{n} \sum_{h=r_n+1}^{n-1} \sum_{s=h}^{h+r_n} (hs)^{-1} \sum_{t=1}^{r_n} [(p_h - p_0^2) + p_0^2] \\ &\leq c \left(\frac{m}{n} m \sum_{h=r_n+1}^{\infty} \xi_h + (mp_0)^2 \frac{r_n}{n} \right) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Next we consider Q_{52} and Q_{53} . By (2.3), we have

$$Q_{52} \leq c \frac{2m^2}{n} \sum_{h=r_n+1}^{n-1} \sum_{s=h}^{n-1} (hs)^{-1} \sum_{t=r_n+1}^n \xi_t \leq c \frac{m \log^2 n}{n} m \sum_{t=r_n+1}^{\infty} \xi_t.$$

The right-hand side converges to zero by using the assumption $m \log^2 n/n = O(1)$ and the condition (2.1). Similarly, using (2.3), we obtain

$$Q_{53} \leq c \frac{m}{n} m \sum_{h=r_n+1}^{\infty} \xi_h.$$

We conclude that $\text{var}(I_{22}) \rightarrow 0$ as $n \rightarrow \infty$.

We proved above that $E(J_{n,A} - J_A(g))^2 \rightarrow 0$, hence $J_{n,A}(g) \xrightarrow{P} J_A(g)$, combined with (2.7), yields $J_{n,A}^\circ(g) \xrightarrow{P} J_A^\circ(g)$.

6. PROOF OF THEOREM 15

We start by proving (3.3). An application of the continuous mapping theorem in $\mathbb{C}(\Pi)$ and Lemma 4 yield in $\mathbb{C}(\Pi)$ for every $k \geq 1$,

$$\left(\frac{m}{n}\right)^{0.5} (\psi_0 (\tilde{\gamma}_A(0) - E\tilde{\gamma}_A(0)) + 2 \sum_{h=1}^k \psi_h (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h))) \xrightarrow{d} \psi_0 Z_0 + 2 \sum_{h=1}^k \psi_h Z_h.$$

Here (Z_h) is a mean zero Gaussian process with covariance structure specified in Lemma 4. In view of Theorem 2 in Dehling et al. [12] relation (3.3) will follow if we can prove the following result.

Lemma 23. *Assume that the conditions of Theorem 15 hold. Then for any $\varepsilon > 0$,*

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left((n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} \psi_h(\lambda) (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \right| > \varepsilon\right) = 0.$$

Proof of Lemma 23. We borrow the techniques of the proof of Theorem 3.2 in Klüppelberg and Mikosch [19]. Without loss of generality we assume that $k = 2^a - 1$ and $n = 2^{b+1}$ where $a < b$ are integers; if k or n do not have this representation we have to modify the proof slightly but we omit details. For integer $q > 0$ and some constant $\kappa > 0$ to be chosen later, let $\varepsilon_q = 2^{-2q/\kappa}$. We have for $\varepsilon > 0$,

$$\begin{aligned} Q &= P\left((n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda) \right| > \varepsilon\right) \\ &\leq P\left((n/m)^{0.5} \sum_{q=a}^b \sup_{\lambda \in \Pi} \left| \sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda) \right| > \varepsilon\right) \\ &\leq P\left(\sum_{q=a}^b \varepsilon_q > \varepsilon\right) + P\left(\bigcup_{q=a}^b \left\{ (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda) \right| > \varepsilon_q \right\}\right) \\ &\leq \sum_{q=a}^b P\left((n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda) \right| > \varepsilon_q\right) = \sum_{q=a}^b Q_q. \end{aligned}$$

In the last steps we used that $P(\sum_{q=a}^b \varepsilon_q > \varepsilon)$ vanishes for fixed ε and sufficiently large a . Next we will bound the expressions Q_q . Write $J_{q,v} = \{(v-1)2^q + 1, \dots, v2^q\}$ and for $j \in J_{q,v}$ and

$$\lambda \in [0, 2^{-2q}\pi],$$

$$Y_{qj}(\lambda) = (n/m)^{0.5} \sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda + (j-1)\pi 2^{-2q}).$$

Then

$$\begin{aligned} Q_q &= P\left((n/m)^{0.5} \max_{v=1, \dots, 2^q} \max_{j \in J_{q,v}} \sup_{\lambda \in [(j-1)\pi 2^{-2q+1}, j\pi 2^{-2q+1}]} \right. \\ &\quad \left. \left| \sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) \psi_h(\lambda) \right| > \varepsilon_q\right) \\ &\leq \sum_{v=1}^{2^q} P\left((n/m)^{0.5} \max_{j \in J_{q,v}} \sup_{\lambda \in [0, 2^{-2q+1}\pi]} |Y_{qj}(\lambda)| > \varepsilon_q\right) = \sum_{v=1}^{2^q} Q_{qv}. \end{aligned}$$

We will bound each of the terms Q_{qv} by twice applying the maximal inequality of Theorem 10.2 in Billingsley [1]. For this reason we have to control the variance of the increments of the process Y_{qj} both as a function of λ and j . In particular, we will derive the following bound

$$(6.1) \quad \frac{n}{m} E\left(\sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) d_h(\omega, \lambda, j, j')\right)^2 \leq c |j - j'|^2 |\lambda - \omega|^{2\beta} K_{k,n},$$

where β is the Hölder coefficient of the function g ,

$$K_{k,n} \leq c \left[m \sum_{h=r_n+1}^{\infty} \xi_h + m \sum_{h=k+1}^{r_n} p_h + r_n/m \right]$$

and for $j < j'$ in $J_{q,v}$, $h \in \{2^q, \dots, 2^{q+1}-1\}$ and $\omega < \lambda$ in $[0, 2^{-2q+1}\pi]$,

$$\begin{aligned} (6.2) \quad d_h(\omega, \lambda, j, j') &= (\psi_h(\lambda + (j'-1)\pi 2^{-2q+1}) - \psi_h(\lambda + (j-1)\pi 2^{-2q+1})) \\ &\quad - (\psi_h(\omega + (j'-1)\pi 2^{-2q+1}) - \psi_h(\omega + (j-1)\pi 2^{-2q+1})) \\ &= \int_{\lambda + (j-1)\pi 2^{-2q+1}}^{\lambda + (j'-1)\pi 2^{-2q+1}} g(x) \cos(hx) dx - \int_{\omega + (j-1)\pi 2^{-2q+1}}^{\omega + (j'-1)\pi 2^{-2q+1}} g(x) \cos(hx) dx \\ &= \int_{(j-1)\pi 2^{-2q+1}}^{(j'-1)\pi 2^{-2q+1}} \left(g(x + \lambda) [\cos(h(x + \lambda)) - \cos(h(x + \omega))] \right. \\ &\quad \left. - [g(x + \lambda) - g(x + \omega)] \cos(h(x + \omega)) \right) dx. \end{aligned}$$

Since g is β -Hölder continuous we have

$$\left| \int_{(j-1)\pi 2^{-2q+1}}^{(j'-1)\pi 2^{-2q+1}} [g(x + \lambda) - g(x + \omega)] \cos(h(x + \omega)) dx \right| \leq c(\lambda - \omega)^\beta (j' - j) 2^{-2q}.$$

Similarly,

$$\begin{aligned} &\left| \int_{(j-1)\pi 2^{-2q}}^{(j'-1)\pi 2^{-2q}} g(x + \lambda) [\cos(h(\lambda + x)) - \cos(h(\omega + x))] dx \right| \\ &= \left| \int_{(j-1)\pi 2^{-2q}}^{(j'-1)\pi 2^{-2q}} g(x + \lambda) (2 \sin(h(\lambda - \omega)/2) \sin(h(\lambda + \omega + 2x)/2)) dx \right| \\ &\leq c h(\lambda - \omega) (j' - j) 2^{-2q} \leq c(\lambda - \omega) (j' - j) 2^{-q}. \end{aligned}$$

The last two inequalities yield for a constant c only depending on g ,

$$(6.3) \quad |d_h(\omega, \lambda, j, j')| \leq c|\lambda - \omega|^\beta |j' - j| 2^{-q}.$$

Using this bound, we have

$$(6.4) \quad \begin{aligned} & \frac{n}{m} E \left(\sum_{h=2^q}^{2^{q+1}-1} (\tilde{\gamma}_A(h) - E\tilde{\gamma}_A(h)) d_h(\omega, \lambda, j, j') \right)^2 \\ & \leq c |j - j'|^2 |\lambda - \omega|^{2\beta} 2^{-2q} \frac{n}{m} \sum_{h=2^q}^{2^{q+1}-1} \sum_{s=h}^{2^{q+1}-1} |\text{cov}(\tilde{\gamma}_A(h), \tilde{\gamma}_A(s))|. \end{aligned}$$

In what follows, it will be convenient to write $\sum_{h,l}^{(q)} = \sum_{h=2^q}^{2^{q+1}-1} \sum_{l=0}^{2^{q+1}-h-1}$. In view of Lemma 1 we can bound the last term in (6.4) as follows:

$$\begin{aligned} & \frac{n}{m} \sum_{h,l}^{(q)} |\text{cov}(\tilde{\gamma}_A(h), \tilde{\gamma}_A(h+l))| \\ = & \frac{m}{n} \sum_{h,l}^{(q)} \left| (n-h-l)\Gamma(0, h, 0, h+l) + \sum_{t=1}^{n-h-l-1} (n-h-l-t)\Gamma(0, h, t, t+h+l) \right. \\ & + \sum_{t=1}^{n-h-1} \min(n-h-l, n-h-t)\Gamma(0, h+l, t, t+h) \\ & \left. - (n-h)(n-h-l)(p_h - p_0^2)(p_{h+l} - p_0^2) \right| \\ \leq & m \sum_{h,l}^{(q)} \left[|\Gamma(0, h, 0, h+l)| + \sum_{t=1}^{h+r_n} |\Gamma(0, t, h, t+h+l)| + \sum_{t=1}^{h+l+r_n} |\Gamma(0, h+l, t, t+h)| \right. \\ & + \frac{1}{n} \left| \sum_{t=h+r_n+1}^{n-h-l-1} (n-t-h-l)\Gamma(0, h, t, t+h+l) \right. \\ & + \sum_{t=h+l+r_n+1}^{n-h-1} (n-t-h)\Gamma(0, h+l, t, t+h) \\ & \left. \left. - (n-h)(n-h-l)(p_h - p_0^2)(p_{h+l} - p_0^2) \right| \right] \\ = & W_1 + W_2 + W_3 + W_4. \end{aligned}$$

We will treat two cases of interest for the sums $\sum_{h,l}^{(q)}$: when $2^{q+1} - 1 \leq r_n$ and $2^q > r_n$. If $2^q \leq r_n < 2^{q+1} - 1$ the sums $\sum_{h,l}^{(q)}$ can be split into two sums corresponding to $h \leq r_n$ and $h > r_n$ and these can be treated in a similar fashion.

We start by studying the case $2^{q+1} - 1 \leq r_n$. Then $r_n \geq 2^{q+1} - 1 \geq h \geq 2^q > k$ and consequently $2^{q+1} - h - 1 \leq 2^q$. Thus, $W_1 \leq c 2^q m \sum_{h=k+1}^{r_n} p_h$. The terms W_2, W_3 have a similar structure and can be treated in the same way; we focus on W_2 . Then we get the following bound from Lemma 1

$$W_2 \leq c 2^{2q} \left[m \sum_{h=k+1}^{r_n} p_h + m \sum_{h=r_n+1}^{2r_n} \xi_h + (r_n/m) \right].$$

In view of (2.4), we also have

$$\begin{aligned}
W_4 &\leq \frac{m}{n} \sum_{h,l}^{(q)} \left[\sum_{t=h+r_n+1}^{n-h-l} (n-t-h-l) |\Gamma(0, h, t, t+h+l) - (p_h - p_0^2)(p_{h+l} - p_0^2)| \right. \\
&\quad + \sum_{t=h+r_n+l+1}^{n-h} (n-t-h) |\Gamma(0, h+l, t, t+h) - (p_h - p_0^2)(p_{h+l} - p_0^2)| \\
&\quad \left. + cnr_n |(p_h - p_0^2)(p_{h+l} - p_0^2)| \right] \\
&\leq c2^{2q}m \sum_{h=r_n+1}^n \xi_h + c \frac{r_n}{m} \left(m \sum_{h=k+1}^{r_n} p_h \right)^2.
\end{aligned}$$

Next we assume that $2^q > r_n$. By (2.3) and (2.4),

$$\begin{aligned}
W_1 &\leq m \sum_{h=2^q}^{2^{q+1}-1} \sum_{l=0}^{r_n} |\Gamma(0, 0, h, h+l) - (p_0 - p_0^2)(p_l - p_0^2)| + 2^q m \sum_{l=0}^{r_n} (p_0 - p_0^2)(p_l - p_0^2) \\
&\quad + m \sum_{h=2^q}^{2^{q+1}-1} \sum_{l=r_n+1}^{2^{q+1}-h-1} |\Gamma(0, 0, h, h+l)| \\
&\leq c2^q m \sum_{h=r_n+1}^{\infty} \xi_h + \frac{2^q r_n}{m} (mp_0)^2.
\end{aligned}$$

We again focus on W_2 ; W_3 can be treated in a similar way.

$$\begin{aligned}
W_2 &\leq m \sum_{h=2^q}^{2^{q+1}-1} \left(\sum_{l=1}^{r_n} \left(\sum_{t=1}^{r_n} + \sum_{t=r_n+1}^h + \sum_{t=h+1}^{h+r_n} \right) + \sum_{l=r_n+1}^{2^{q+1}-1} \sum_{t=1}^{h+r_n} \right) |\Gamma(0, t, h, t+h+l)| \\
&\leq c2^{2q}m \sum_{h=r_n+1}^{\infty} \xi_h + c2^{2q} \frac{r_n}{m} (mp_0)^2
\end{aligned}$$

To obtain the bounds for W_4 we use (2.4):

$$\begin{aligned}
W_4 &\leq \frac{m}{n} \sum_{h=2^q}^{2^{q+1}-1} \sum_{l=0}^{2^{q+1}-h-1} \left[\sum_{t=h+r_n+1}^{n-h-l} (n-t-h-l) \right. \\
&\quad \times |\Gamma(0, h, t, t+h+l) - (p_h - p_0^2)(p_{h+l} - p_0^2)| \\
&\quad + \sum_{t=h+r_n+l+1}^{n-h} (n-t-h) |\Gamma(0, h+l, t, t+h) - (p_h - p_0^2)(p_{h+l} - p_0^2)| \\
&\quad \left. + cn2^q |(p_h - p_0^2)(p_{h+l} - p_0^2)| \right] \\
&\leq c2^{2q}m \sum_{t=r_n+1}^{\infty} \xi_t + c(2^q/m) \left(m \sum_{h=r_n+1}^{\infty} \xi_h \right)^2.
\end{aligned}$$

Collecting the bounds for W_i , $i \leq 4$, and using (6.4), we finally proved (6.1).

Using this bound, we can apply the maximal inequality of Theorem 10.2 in Billingsley [1] with respect to the variable $\lambda \leq 2^{-2q}\pi$ and for fixed j, j' :

$$\begin{aligned}
P\left(\max_{0 \leq \lambda \leq 2^{-2q}\pi} |Y_j(\lambda) - Y_{j'}(\lambda)| > \varepsilon_q\right) &\leq c\varepsilon_q^{-2} (2^{-2q}\pi)^{2\beta} (j - j')^2 K_{k,n} \\
&\leq c2^{4q(1-\beta+\kappa^{-1})} ((j - j')2^{-2q})^2 K_{k,n}.
\end{aligned}$$

Another application of this maximal inequality to $\max_{0 \leq \lambda \leq 2^{-2q}\pi} |Y_j(\lambda)|$ with respect to the variable $j \in J_{q,v}$ yields

$$Q_{qv} = P\left(\max_{j \in \{(v-1)2^q+1, \dots, v2^q\}} \max_{0 \leq \lambda \leq 2^{-2q}\pi} |Y_j(\lambda)| > \varepsilon_q\right) \leq c2^{4q(2^{-1}-\beta+\kappa^{-1})} K_{k,n}.$$

Then we also have

$$Q_q \leq \sum_{v=1}^{2^q} Q_{qv} \leq c2^{4q(3/4-\beta+\kappa^{-1})} K_{k,n}.$$

The right-hand side converges to zero as $q \rightarrow \infty$ provided $\beta \in (3/4, 1]$ and κ is chosen sufficiently large. Therefore we conclude for every $\varepsilon > 0$,

$$(6.5) \quad Q \leq \sum_{q=a}^b Q_q \leq cK_{k,n} \sum_{q=a}^{\infty} 2^{4q(3/4-\beta+\kappa^{-1})}.$$

The right-hand side converges to zero by first letting $n \rightarrow \infty$ and then $k \rightarrow \infty$. This concludes the proof of (3.3).

Next we turn to the proof of (3.4). It will follow from (3.3) once we prove the following lemma.

Lemma 24. *Assume that the conditions of Theorem 15 hold. Then for any $\varepsilon > 0$, as $n \rightarrow \infty$,*

$$P\left((n/m)^{0.5} \sup_{\lambda \in \Pi} |(\widehat{J}_{n,A}(\lambda) - E\widehat{J}_{n,A}(\lambda)) - (J_{n,A}(\lambda) - EJ_{n,A}(\lambda))| > \varepsilon\right) \rightarrow 0.$$

Proof of Lemma 24: For any fixed $k \geq 1$ we have

$$\begin{aligned} & P\left((n/m)^{0.5} \sup_{\lambda \in \Pi} |(\widehat{J}_{n,A}(\lambda) - E\widehat{J}_{n,A}(\lambda)) - (J_{n,A}(\lambda) - EJ_{n,A}(\lambda))| > \varepsilon\right) \\ & \leq P\left((n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=0}^k (\widetilde{\gamma}_A(h) - E\widetilde{\gamma}_A(h))(\psi_h(\lambda) - \widehat{\psi}_h(\lambda)) \right| > \varepsilon/3\right) \\ & \quad + P\left((n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^n (\widetilde{\gamma}_A(h) - E\widetilde{\gamma}_A(h))\psi_h(\lambda) \right| > \varepsilon/3\right) \\ & \quad + P\left((n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^n (\widetilde{\gamma}_A(h) - E\widetilde{\gamma}_A(h))\widehat{\psi}_h(\lambda) \right| > \varepsilon/3\right) \\ & = V_1 + V_2 + V_3. \end{aligned}$$

An application of Chebyshev's and Hölder's inequalities yields,

$$\begin{aligned} V_1 & \leq 9\varepsilon^{-2} \frac{n}{m} E \sup_{\lambda \in \Pi} \left| \sum_{h=0}^k (\widetilde{\gamma}_A(h) - E\widetilde{\gamma}_A(h))(\psi_h(\lambda) - \widehat{\psi}_h(\lambda)) \right|^2 \\ & \leq c \frac{n}{m} E \sup_{\lambda \in \Pi} \sum_{h=0}^k (\widetilde{\gamma}_A(h) - E\widetilde{\gamma}_A(h))^2 |\psi_h(\lambda) - \widehat{\psi}_h(\lambda)| \sum_{s=0}^k |\psi_s(\lambda) - \widehat{\psi}_s(\lambda)| \\ & \leq c k \frac{n}{m} \sum_{h=0}^k \text{var}(\widetilde{\gamma}_A(h)) \sup_{x \in \Pi} |\psi_h(x) - \widehat{\psi}_h(x)|. \end{aligned}$$

Next we will study $\sup_{\lambda \in \Pi} |\psi_h(\lambda) - \widehat{\psi}_h(\lambda)|$. Trivially, for $x \in \Pi$,

$$\left| \int_{\omega_n(x_n)}^x \cos(h\lambda) g(\lambda) d\lambda \right| \leq c/n,$$

where the constant c only depends on g . We also have for the frequencies $x \in \Pi$,

$$\begin{aligned}
& |\psi_h(\omega_n(x_n)) - \widehat{\psi}_h(\omega_n(x_n))| \\
&= \left| \sum_{i=1}^{x_n} \left(\int_{\omega_n(i-1)}^{\omega_n(i)} \cos(h\lambda) g(\lambda) d\lambda - \omega_n(1) \cos(h\omega_n(i)) g(\omega_n(i)) \right) \right| \\
(6.6) \quad &\leq \sum_{i=1}^{x_n} \left| \int_{\omega_n(i-1)}^{\omega_n(i)} \cos(h\lambda) (g(\lambda) - g(\omega_n(i))) d\lambda \right| \\
&\quad + \left| \sum_{i=1}^{x_n} g(\omega_n(i)) \left(\frac{\sin(h\omega_n(i)) - \sin(h\omega_n(i-1))}{h} - \omega_n(1) \cos(h\omega_n(i)) \right) \right|.
\end{aligned}$$

(6.7)

Since g is β -Hölder continuous there exists a constant $c > 0$ such that

$$|g(\lambda) - g(\omega_n(i))| \leq cn^{-\beta}, \quad \lambda \in [\omega_n(i-1), \omega_n(i)].$$

Therefore the term in (6.6) is bounded by $cn^{-\beta}$. A Taylor expansion as $z \rightarrow 0$ yields $\sin(z) = z - z^3/3! + o(z^3)$. Then we have for $h \leq n$,

$$\begin{aligned}
& \left| \frac{\sin(h\omega_n(i)) - \sin(h\omega_n(i-1))}{h} - \omega_n(1) \cos(h\omega_n(i)) \right| \\
&= \left| 2h^{-1} \sin(h\omega_n(0.5)) \cos(h\theta(i+0.5)) - \omega_n(1) \cos(h\omega_n(i)) \right| \\
&= \left| 2h^{-1} (\sin(h\omega_n(0.5)) - h\omega_n(0.5)) \cos(h\theta(i+0.5)) \right. \\
&\quad \left. + \omega_n(1) (\cos(h\omega_n(i+0.5)) - \cos(h\omega_n(i))) \right| \\
&\leq c(h\omega_n(1))^3 + \omega_n(1) \left| 2 \sin(h\omega_n(0.25)) \sin(h\omega_n(i+0.25)) \right| \leq c(h^3 n^{-3} + hn^{-2}).
\end{aligned}$$

Consequently, we have the bound $c(k/n)(1 + k^2/n)$ for (6.7) uniformly for $x \in \Pi$ and $h \leq k$. Thus, uniformly for $h \leq k$,

$$\sup_{x \in \Pi} |\widetilde{\psi}_h(x) - \widehat{\psi}_h(x)| \leq c[n^{-\beta} + (k/n)(1 + k^2/n)].$$

As we have shown in Lemma 4, $(n/m) \sum_{h=0}^k \text{var}(\widetilde{\gamma}_A(h)) \leq ck$; see also Davis and Mikosch [7], Lemma 5.2. Thus, as $n \rightarrow \infty$,

$$V_1 \leq c[k^2 n^{-\beta} + (k^3/n)(1 + k^2/n)] \rightarrow 0.$$

It follows from Lemma 23 that $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} V_2 = 0$. We adapt the proof of Lemma 23 for the case V_3 . Abusing notation, consider

$$\begin{aligned}
d_h(\omega, \lambda, j, j') &= (\widehat{\psi}_h(\lambda + (j' - 1)\pi 2^{-2q+1}) - \widehat{\psi}_h(\lambda + (j - 1)\pi 2^{-2q+1})) \\
&\quad - (\widehat{\psi}_h(\omega + (j' - 1)\pi 2^{-2q+1}) - \widehat{\psi}_h(\omega + (j - 1)\pi 2^{-2q+1})).
\end{aligned}$$

Recall that we assume $n = 2^b$ for some integer b and $x_n = [n\lambda/(2\pi)]$. Therefore for $\lambda \in \Pi$ and integer j ,

$$\begin{aligned}
(\lambda + (j - 1)\pi 2^{-2q+1})_n &= [n\lambda/(2\pi) + (j - 1)2^{-2q+b}] \\
&= [n\lambda/(2\pi)] + (j - 1)2^{-2q+b} \\
&= \lambda_n + (j - 1)2^{-2q+b}.
\end{aligned}$$

Thus we can write

$$\begin{aligned}
& d_h(\omega, \lambda, j, j') \\
&= \frac{2\pi}{n} \sum_{i=\lambda_n+(j-1)2^{b-2q}}^{\lambda_n+(j'-1)2^{b-2q}} g(\omega_n(i)) \cos(h\omega_n(i)) \\
&\quad - \frac{2\pi}{n} \sum_{i=\omega_n+(j-1)2^{b-2q}}^{\omega_n+(j'-1)2^{b-2q}} g(\omega_n(i)) \cos(h\omega_n(i)) \\
&= \frac{2\pi}{n} \sum_{i=(j-1)2^{b-2q}}^{(j'-1)2^{b-2q}} \left(g(\omega_n(\lambda_n+i)) [\cos(h\omega_n(\lambda_n+i)) - \cos(h\omega_n(\omega_n+i))] \right. \\
&\quad \left. - [g(\omega_n(\lambda_n+i)) - g(\omega_n(\omega_n+i))] \cos(h\omega_n(\omega_n+i)) \right) = T_1 + T_2.
\end{aligned}$$

Calculation yields

$$\begin{aligned}
|T_1| &\leq c|\omega_n(\lambda_n) - \omega_n(\omega_n)| |(j' - j)2^{-2q}| 2^q \leq c|(\lambda_n - \omega_n)/n| |(j' - j)2^{-2q}| 2^q, \\
|T_2| &\leq c|\omega_n(\lambda_n) - \omega_n(\omega_n)|^\beta |(j' - j)2^{-2q}| \leq c|(\lambda_n - \omega_n)/n|^\beta |(j' - j)2^{-2q}|.
\end{aligned}$$

Combining these bounds, we have,

$$|d_h(\omega, \lambda, j, j')| \leq c|(\lambda_n - \omega_n)/n|^\beta |(j' - j)2^{-2q}| 2^q.$$

In the remaining argument we can follow the proof of Lemma 23; the only difference is that we have to replace the supremum over $\lambda, \omega \in [0, j2^{-2q+1}]$ by the corresponding quantities $\lambda_n/n, \omega_n/n \in [0, j2^{-2q+1}]$. This proves

$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} V_3 = 0$ and concludes the proof of the lemma.

The proofs of (3.6) and (3.7) are completely analogous. Instead of the relations (2.8) one has to use (2.10).

7. PROOF OF THEOREM 17

We adapt the proof of Theorem 15. We need to prove that

$$n \sum_{h,l}^{(q)} |\text{cov}(\tilde{\gamma}_A(h), \tilde{\gamma}_A(h+l))| \leq c2^q,$$

where $\sum_{h,l}^{(q)}$ is defined in the proof of Lemma 23. Here $h > \eta$.

$$\begin{aligned}
& n \sum_{h,l}^{(q)} |\text{cov}(\tilde{\gamma}_A(h), \tilde{\gamma}_A(h+l))| \\
&= \frac{m^2}{n} \sum_{h,l}^{(q)} \left| (n-h-l)\Gamma(0, h, 0, h+l) + \sum_{t=1}^{n-h-l-1} (n-h-l-t)\Gamma(0, h, t, t+h+l) \right. \\
&\quad \left. + \sum_{t=1}^{n-h-1} \min(n-h-l, n-h-t)\Gamma(0, h+l, t, t+h) \right| \\
&\leq m^2 \sum_{h=2^q}^{2^{q+1}-1} |\Gamma(0, 0, h, h)| + m^2 \sum_{h=2^q}^{2^{q+1}-1} \sum_{l=1}^{2^{q+1}-h-1} |\Gamma(0, 0, h, h+l)| \\
&\quad + m^2 \sum_{h,l}^{(q)} \sum_{t=1}^{n-h-l-1} |\Gamma(0, h, t, t+h+l)| + m^2 \sum_{h,l}^{(q)} \sum_{t=1}^{n-h-l-1} |\Gamma(0, h+l, t, t+h)| \\
&= m^2 \sum_{h=2^q}^{2^{q+1}-1} |\Gamma(0, 0, h, h)| + m^2 \sum_{h=2^q}^{2^{q+1}-1} \sum_{l=1}^{\eta} |\Gamma(0, 0, h, h+l)| \\
&\quad + m^2 \sum_{h=2^q}^{2^{q+1}-1} \sum_{l=1}^{\eta} \sum_{t=1}^{\eta} |\Gamma(0, h, t, t+h+l)| + m^2 \sum_{h=2^q}^{2^{q+1}-1} \sum_{l=1}^{\eta} \sum_{t=1}^{\eta} |\Gamma(0, h+l, t, t+h)| \\
&\leq c2^q
\end{aligned}$$

In the above calculation, we use the facts that for $s \leq t \leq u \leq v$, $\Gamma(s, t, u, v) = 0$ where $t - s > \eta$ or $v - u > \eta$.

In the remaining argument we can follow the proof of Theorem 15; instead of Lemma 4 we use the central limit theory of Lemma 6. \square

8. PROOF OF THEOREM 21

We will mimic the proof of Theorem 15. We start by proving a result for the bootstrapped sample extremogram $\hat{\gamma}_A^*$ analogous to Theorem 20.

Lemma 25. *Under the conditions and with the notation of Theorem 20, for $h \geq 0$,*

$$d\left((n/m)^{0.5}(\hat{\gamma}_A^*(i) - E^*\hat{\gamma}_A^*(i))_{i=0,\dots,h}, (Z_i)_{i=0,\dots,h}\right) \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

Proof. We start by observing (see Lemma 27) that for $h \geq 0$

$$\begin{aligned}
E^*\hat{\gamma}_A^*(h) &= \frac{m}{n} (n-h) E^* \tilde{I}_1^* \tilde{I}_{1^*+h} \\
&= (1-h/n) \left[\tilde{\gamma}_A(h) + \frac{m}{n} \sum_{t=n-h+1}^n \tilde{I}_t \tilde{I}_{t+h} \right], \\
E^*\hat{\gamma}_A^*(h) &= \frac{m}{n} (n-h) E^* \hat{I}_1^* \hat{I}_{(1+h)^*} \\
&= (1-h/n) (1-\theta)^h \left[\hat{\gamma}_A(h) + \frac{m}{n} \sum_{t=n-h+1}^n \hat{I}_t \hat{I}_{t+h} \right],
\end{aligned}$$

where we interpret indices larger than n modulo n , and therefore

$$(8.1) \quad (n/m)^{0.5} [(1-\theta)^h E^*\hat{\gamma}_A^*(h) - E^*\hat{\gamma}_A^*(h)] = O_P(m^{-1}) \xrightarrow{P} 0,$$

where we used that $\bar{T}_n^2 - p_0^2 = O_P(1/\sqrt{mn})$. By virtue of Theorem 20 it suffices to show that for any $\varepsilon > 0$ and $h \geq 0$, as $n \rightarrow \infty$,

$$P^*\left((n/m)^{0.5} \left| (1-\theta)^h (\tilde{\gamma}_A^*(h) - E^* \tilde{\gamma}_A^*(h)) - (\hat{\gamma}_A^*(h) - E^* \hat{\gamma}_A^*(h)) \right| > \varepsilon\right) \xrightarrow{P} 0.$$

Markov's inequality ensures that it suffices to prove that

$$\frac{n}{m} \text{var}^*((1-\theta)^h \tilde{\gamma}_A^*(h) - \hat{\gamma}_A^*(h)) \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

We observe that

$$\begin{aligned} & \frac{n}{m} \text{var}^*((1-\theta)^h \tilde{\gamma}_A^*(h) - \hat{\gamma}_A^*(h)) \\ &= m \left(1 - \frac{h}{n}\right) \text{var}^*\left(\hat{I}_1^* \hat{I}_{(1+h)}^* - (1-\theta)^h \tilde{I}_1^* \tilde{I}_{1^*+h}\right) \\ & \quad + 2m \sum_{s=1}^{n-h-1} \left(1 - \frac{h+s}{n}\right) \times \\ & \quad \text{cov}^*\left(\hat{I}_1^* \hat{I}_{(1+h)}^* - (1-\theta)^h \tilde{I}_1^* \tilde{I}_{1^*+h}, \hat{I}_{(1+s)}^* \hat{I}_{(1+s+h)}^* - (1-\theta)^h \tilde{I}_{(1+s)}^* \tilde{I}_{(1+s)^*+h}\right) \\ &= m \left(1 - \frac{h}{n}\right) \text{var}^*\left(\hat{I}_1^* \hat{I}_{(1+h)}^* - (1-\theta)^h \tilde{I}_1^* \tilde{I}_{1^*+h}\right) \\ & \quad + 2m \sum_{s=1}^{n-h-1} \left(1 - \frac{h+s}{n}\right) \left[\text{cov}^*(\hat{I}_1^* \hat{I}_{(1+h)}^*, \hat{I}_{(1+s)}^* \hat{I}_{(1+s+h)}^*) \right. \\ & \quad - (1-\theta)^h \text{cov}^*(\hat{I}_1^* \hat{I}_{(1+h)}^*, \tilde{I}_{(1+s)}^* \tilde{I}_{(1+s)^*+h}) \\ & \quad - (1-\theta)^h \text{cov}^*(\tilde{I}_1^* \tilde{I}_{1^*+h}, \hat{I}_{(1+s)}^* \hat{I}_{(1+s+h)}^*) \\ & \quad \left. + (1-\theta)^{2h} \text{cov}^*(\tilde{I}_1^* \tilde{I}_{1^*+h}, \tilde{I}_{(1+s)}^* \tilde{I}_{(1+s)^*+h}) \right] = Q_1 + Q_2. \end{aligned}$$

We will show that the right-hand side converges to zero in P -probability, where we focus on Q_2 and omit the details for Q_1 . We start by looking at the summands in Q_2 for fixed $s \leq h$, using the structure of the covariances in Lemma 27. The expressions for the covariances in Lemma 27 contain terms with normalization n^{-2} . For example, by (8.7) a corresponding term in Q_2 is of the order

$$m \left(n^{-1} \sum_{i=1}^n \tilde{I}_i \tilde{I}_{i+h} \right)^2 = m^{-1} \left(\frac{m}{n} \sum_{i=1}^n \tilde{I}_i \tilde{I}_{i+h} \right)^2 = O_P(m^{-1}),$$

since $\frac{m}{n} \sum_{i=1}^n \tilde{I}_i \tilde{I}_{i+h} \xrightarrow{P} \gamma_A(h)$; see Lemma 4. In the latter sums, the \tilde{I}_i 's can be exchanged by the I_i 's or the \hat{I}_i 's. Therefore all other terms in Q_2 with normalization mn^{-2} converge to zero in P -probability. Another appeal to Lemma 27 shows that it remains to consider those expressions in Q_2 that are normalized by mn^{-1} again for fixed $s \leq h$. From (8.9) and (8.10) we see that, on one hand, we have to deal with the differences

$$(8.2) \quad (1-\theta)^{s+h} \frac{m}{n} \sum_{i=1}^n \hat{I}_i \hat{I}_{i+s} \hat{I}_{i+h} \hat{I}_{i+s+h} - (1-\theta)^{s+2h} \frac{m}{n} \sum_{i=1}^n \tilde{I}_i \tilde{I}_{i+h} \hat{I}_{i+s} \hat{I}_{i+s+h},$$

but both sums are consistent estimators of $\lim_{n \rightarrow \infty} mP(a_m^{-1}X_0 \in A, a_m^{-1}X_s \in A, a_m^{-1}X_h \in A, a_m^{-1}X_{s+h} \in A)$ (see [7], Theorem 3.1). Therefore (8.2) converges to zero in P -probability. On the other hand, in view of (8.7) and (8.8) we have to deal with the differences, for $s \leq h$,

$$(1-\theta)^{s+2h} \frac{m}{n} \sum_{i=1}^n \tilde{I}_i \tilde{I}_{i+s} \tilde{I}_{i+h} \tilde{I}_{i+s+h} - (1-\theta)^{2h} \frac{m}{n} \sum_{i=1}^n \hat{I}_i \hat{I}_{i+h} \tilde{I}_{i+s} \tilde{I}_{i+s+h},$$

which again converge to zero in P -probability. These arguments finish the proof for $s \leq h$.

An inspection of the covariances in Lemma 27 shows that for $s > h$ all expressions with normalization n^{-2} do not depend on s . The corresponding aggregated terms in Q_2 are then given by

$$\begin{aligned}
& 2m \sum_{s=h+1}^{n-h-1} \left(1 - \frac{h+s}{n}\right) \left[- (1-\theta)^{s+h} \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right)^2 \right. \\
& + (1-\theta)^{s+h} \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right) \left(n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} \right) \\
& \left. + (1-\theta)^{s+2h} \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right) \left(n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} \right) - (1-\theta)^{s+2h} \left(n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} \right)^2 \right] \\
& = -2m^{-1} \left(\frac{m}{n} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} - \frac{m}{n} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} \right) \left(\frac{m}{n} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right) \sum_{s=h+1}^{n-h-1} \left(1 - \frac{h+s}{n}\right) (1-\theta)^{s+h} \\
& - 2m^{-1} \left(\frac{m}{n} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} - \frac{m}{n} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right) \left(\frac{m}{n} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} \right) \sum_{s=h+1}^{n-h-1} \left(1 - \frac{h+s}{n}\right) (1-\theta)^{s+2h} \\
& = O_P(1/(\theta\sqrt{mn})) = o_P(1).
\end{aligned}$$

In the last step we used (8.1) and the assumption $n\theta^2/m \rightarrow \infty$. Finally, we deal with the remaining terms in Q_2 . In view of Lemma 27 they are given by

$$\begin{aligned}
& 2m \sum_{s=h+1}^{n-h-1} \left(1 - \frac{h+s}{n}\right) \left[(1-\theta)^{s+h} n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+s} \widehat{I}_{i+h} \widehat{I}_{i+s+h} \right. \\
& - (1-\theta)^{s+h} n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \widetilde{I}_{i+s} \widetilde{I}_{i+s+h} \\
& \left. - (1-\theta)^{s+2h} n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} \widehat{I}_{i+s} \widehat{I}_{i+s+h} + (1-\theta)^{s+2h} n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+s} \widetilde{I}_{i+h} \widetilde{I}_{i+s+h} \right] \\
& = 2m \sum_{s=h+1}^{n-h-1} \left(1 - \frac{h+s}{n}\right) (1-\theta)^{s+h} n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} (\widehat{I}_{i+s} \widehat{I}_{i+s+h} - \widetilde{I}_{i+s} \widetilde{I}_{i+s+h}) \\
& + 2m \sum_{s=h+1}^{n-h-1} \left(1 - \frac{h+s}{n}\right) (1-\theta)^{s+2h} n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} (\widetilde{I}_{i+s} \widetilde{I}_{i+s+h} - \widehat{I}_{i+s} \widehat{I}_{i+s+h}) = J_0.
\end{aligned}$$

Using the assumption $n\theta^2/m \rightarrow \infty$, we have

$$\begin{aligned}
E|J_0| & \leq cm \sum_{s=h+1}^{n-h-1} (1-\theta)^{s+h} E|\widehat{I}_0 \widehat{I}_h - \widetilde{I}_0 \widetilde{I}_h| \\
& \leq cm E|p_0 - \bar{I}_n| \sum_{s=h+1}^{n-h-1} (1-\theta)^{s+h} \leq c(m/n)^{0.5} \theta^{-1} = o(1).
\end{aligned}$$

This finishes the proof of the lemma. \square

We conclude from Lemma 25 that for any $k \geq 1$, as $n \rightarrow \infty$,

$$\begin{aligned}
& d\left((n/m)^{0.5} \left(\psi_0(\widehat{\gamma}_A^*(0) - E^* \widehat{\gamma}_A^*(0)) + 2 \sum_{h=1}^k \psi_h(\widehat{\gamma}_A^*(h) - E^* \widehat{\gamma}_A^*(h)) \right), \right. \\
& \left. \psi_0 Z_0 + 2 \sum_{h=1}^k \psi_h Z_h \right) \xrightarrow{P} 0,
\end{aligned}$$

where the dependence structure of (Z_h) is defined in Lemma 4.

The proof of the theorem is finished by the following result which parallels Lemma 23.

Lemma 26. *Assume the conditions of Theorem 21. Then the following relation holds for $\delta > 0$*

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left((n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} \psi_h(\lambda) (\hat{\gamma}_A^*(h) - E^* \hat{\gamma}_A^*(h)) \right| > \delta\right) = 0. \quad (8.3)$$

Proof. We follow the lines of the proof of Lemma 23 and use the same notation. We again assume without loss of generality that $k = 2^a - 1$ and $n = 2^{b+1}$ for integers $a < b$, a chosen sufficiently large, and we write $\varepsilon_q = 2^{-2q/\kappa}$ for $\kappa > 0$ to be chosen later. Then, for large a depending on $\varepsilon > 0$, the steps of the proof lead to the inequality (cf. (6.5))

$$\begin{aligned} Q^* &= P^*\left((n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} (\hat{\gamma}_A^*(h) - E^* \hat{\gamma}_A^*(h)) \psi_h(\lambda) \right| > \varepsilon\right) \\ &\leq c \sum_{q=a}^b 2^{4q(0.75-\beta+\kappa^{-1})} K_q, \end{aligned}$$

where $\beta \in (3/4, 1]$ is the Hölder coefficient of the function g , the number $\kappa > 0$ can be chosen arbitrarily large and

$$K_q = \frac{n}{m} \sum_{h=2^q}^{2^{q+1}-1} \sum_{s=h}^{2^{q+1}-1} |\text{cov}^*(\hat{\gamma}_A^*(h), \hat{\gamma}_A^*(s))|.$$

By the Cauchy-Schwarz inequality, for $s, h \in [2^q, 2^{q+1})$ and $h \leq s$,

$$(n/m)^2 |\text{cov}^*(\hat{\gamma}_A^*(h), \hat{\gamma}_A^*(s))|^2 \leq (n/m) \text{var}^*(\hat{\gamma}_A^*(h)) (n/m) \text{var}^*(\hat{\gamma}_A^*(s)).$$

We will show that

$$(8.4) \quad (n/m) E \text{var}^*(\hat{\gamma}_A^*(h)) \leq c$$

for some constant c , uniformly for $k \leq h \leq n$ and n . Then

$$EQ^* \leq c \sum_{q=a}^b 2^{4q(3/4-\beta+\kappa^{-1})} \leq c \sum_{q=a}^{\infty} 2^{4q(3/4-\beta+\kappa^{-1})}.$$

The right-hand side converges since $\beta \in (3/4, 1]$ and κ can be chosen arbitrarily large. Moreover, the right-hand side converges to zero as $k \rightarrow \infty$.

Thus it remains to show (8.4). In view of Lemma 27 we have

$$\begin{aligned}
& (n/m)E\text{var}^*(\hat{\gamma}_A^*(h)) \\
&= (m/n)\left[(n-h)E\text{var}^*(\hat{I}_1^*\hat{I}_{(1+h)}^*)\right. \\
&\quad \left.+2\sum_{t=1}^{n-h-1}(n-h-t)E\text{cov}^*(\hat{I}_1^*\hat{I}_{(1+h)}^*, \hat{I}_{(1+t)}^*\hat{I}_{(1+t+h)}^*)\right] \\
&= \left[m(1-h/n)(1-\theta)^{2h}\left[E(\hat{I}_1\hat{I}_{1+h})^2 - E\left(n^{-1}\sum_{i=1}^n\hat{I}_i\hat{I}_{i+h}\right)^2\right]\right] \\
&\quad +2m\sum_{t=1}^{n-h-1}(1-(h+t)/n)\left[n^{-1}\sum_{i=1}^nE\hat{I}_i\hat{I}_{i+h}\hat{I}_{i+t}\hat{I}_{i+t+h}\right](1-\theta)^{t+h} \\
&\quad +2m\sum_{t=1}^{\min(h-1, n-h-1)}(1-(h+t)/n)E\left(n^{-1}\sum_{i=1}^n\hat{I}_i\hat{I}_{i+t}\right)^2((1-\theta)^{2t} - (1-\theta)^{t+h}) \\
&\quad -2m\sum_{t=1}^{\min(h-1, n-h-1)}(1-(h+t)/n)E\left(n^{-1}\sum_{i=1}^n\hat{I}_i\hat{I}_{i+h}\right)^2(1-\theta)^{2h} \\
&\quad -2m\sum_{t=h}^{n-h-1}(1-(h+t)/n)E\left(n^{-1}\sum_{i=1}^n\hat{I}_i\hat{I}_{i+h}\right)^2(1-\theta)^{t+h} \\
&\leq mE(\hat{I}_1\hat{I}_{1+h})^2 + 2m\sum_{t=1}^{n-h-1}(1-(h+t)/n)\left(n^{-1}\sum_{i=1}^nE\hat{I}_i\hat{I}_{i+h}\hat{I}_{i+t}\hat{I}_{i+t+h}\right)(1-\theta)^{t+h} \\
&\quad +2m\sum_{t=1}^{\min(h-1, n-h-1)}(1-(h+t)/n)E\left(n^{-1}\sum_{i=1}^n\hat{I}_i\hat{I}_{i+t}\right)^2(1-\theta)^{2t} \\
&= V_1 + V_2 + V_3.
\end{aligned}$$

We observe that, for some constant $c_0 > 0$,

$$V_1 \leq mE(\hat{I}_1\hat{I}_{1+h})^2 \leq cm\left[EI_1I_{1+h} + (E\bar{I}_n)^2\right] \leq cm p_0 \leq c_0.$$

For V_2 , we observe that for $i \leq n$,

$$\begin{aligned}
& m\theta^{-1}\left|E\left[\hat{I}_i\hat{I}_{i+h}\hat{I}_{i+t}\hat{I}_{i+t+h} - \tilde{I}_i\tilde{I}_{i+h}\tilde{I}_{i+t}\tilde{I}_{i+t+h}\right]\right| \\
&\leq cm\theta^{-1}E|\bar{I}_n - p_0| = O(\sqrt{m/n}\theta^{-1}) = o(1),
\end{aligned}$$

by virtue of the condition $n\theta^2/m \rightarrow \infty$. Therefore, for showing that $|V_2| \leq c$ uniformly for h, n , it suffices to show that $|\tilde{V}_2| \leq c$, where \tilde{V}_2 is obtained from V_2 by replacing the \hat{I}_t 's by the corresponding \tilde{I}_t 's. Taking into account $E\tilde{I}_1\tilde{I}_{1+t} = p_t - p_0^2$ and the Cauchy-Schwarz inequality, we have for a fixed integer $M > 0$,

$$\begin{aligned}
|\tilde{V}_2| &\leq cm\sum_{t=1}^{n-h-1}\left|n^{-1}\sum_{i=1}^nE\tilde{I}_i\tilde{I}_{i+h}\tilde{I}_{i+t}\tilde{I}_{i+t+h}\right| \\
&= cm\sum_{t=1}^{n-h-1}|E\tilde{I}_1\tilde{I}_{1+h}\tilde{I}_{1+t}\tilde{I}_{1+t+h}| \\
&\leq (mp_0)M + cm\sum_{t=M+1}^{r_n}(p_t + p_0^2) + cm\sum_{t=r_n+1}^{\infty}\xi_t \leq c,
\end{aligned}$$

in view of condition (M) and regular variation. A similar argument as for V_2 shows that one may replace the \widehat{I}_t 's in V_3 by the corresponding \widetilde{I}_t 's. We denote the resulting quantity by \widetilde{V}_3 . Then we have

$$\begin{aligned}
\widetilde{V}_3 &\leq m \sum_{t=1}^n (1-\theta)^t E \left(n^{-1} \sum_{i=1}^{n-t} \widetilde{I}_i \widetilde{I}_{i+t} + n^{-1} \sum_{i=n-t+1}^n \widetilde{I}_i \widetilde{I}_{i+t-n} \right)^2 \\
&\leq cm \sum_{t=1}^n (1-\theta)^t E \left(n^{-1} \sum_{i=1}^{n-t} \widetilde{I}_i \widetilde{I}_{i+t} \right)^2 + cm \sum_{t=1}^n (1-\theta)^t E \left(n^{-1} \sum_{i=n-t+1}^n \widetilde{I}_i \widetilde{I}_{i+t-n} \right)^2 \\
&= \widetilde{V}_{31} + \widetilde{V}_{32}.
\end{aligned}$$

We will only deal with \widetilde{V}_{31} , the other term can be bounded in a similar way. We observe that for fixed $M > 1$, using condition (M),

$$\begin{aligned}
\widetilde{V}_{31} &\leq c \frac{m}{n} \sum_{t=1}^n (1-\theta)^t \left(E(\widetilde{I}_1 \widetilde{I}_{1+t})^2 + 2 \sum_{s=1}^{n-t-1} |E \widetilde{I}_1 \widetilde{I}_{1+t} \widetilde{I}_{1+s} \widetilde{I}_{1+s+t}| \right) \\
&\leq o(1) + c \frac{m}{n} \sum_{t=1}^n (1-\theta)^t \sum_{s=1}^{n-t-1} |E \widetilde{I}_1 \widetilde{I}_{1+t} \widetilde{I}_{1+s} \widetilde{I}_{1+s+t}| \\
&\leq o(1) + c \frac{m}{n} \sum_{t=1}^n (1-\theta)^t \sum_{s=M+1}^{r_n} (p_s + p_0^2) + c \frac{m}{n} \sum_{t=1}^n (1-\theta)^t \sum_{r_n+1 \leq s \leq n-t-1, s \leq t} \xi_s \\
&\quad + c \frac{m}{n} \sum_{t=1}^n (1-\theta)^t \sum_{r_n+1 \leq s \leq n-t-1, s > t} (|E \widetilde{I}_1 \widetilde{I}_{1+t} \widetilde{I}_{1+s} \widetilde{I}_{1+s+t} - (p_t - p_0^2)^2| + (p_t - p_0^2)^2).
\end{aligned}$$

In view of condition (M), the first two terms on the right-hand side are negligible as $n \rightarrow \infty$. The third term is bounded by

$$c \frac{m}{n} \sum_{t=1}^n (1-\theta)^t \sum_{r_n+1 \leq s \leq n-t-1, s > t} \xi_{s-t} + cm \sum_{t=1}^n (1-\theta)^t (p_t - p_0^2)^2.$$

Multiple use of (M) again shows that the right-hand side is negligible. This proves (8.4). $\square \quad \square$

Lemma 27. *Under the conditions of Theorem 21 the following relations hold for $s, h \geq 0$:²*

$$(8.5) \quad E^* \widehat{I}_{1*} = 0,$$

$$(8.6) \quad E^* \widehat{I}_{1*} \widehat{I}_{(1+h)*} = (1 - \theta)^h n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h}, \quad E^* \widetilde{I}_{1*} \widetilde{I}_{1*+h} = n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h},$$

$$(8.7) \quad \text{cov}^*(\widetilde{I}_{1*} \widetilde{I}_{1*+h}, \widetilde{I}_{(1+s)*} \widetilde{I}_{(1+s)*+h}) \\ = (1 - \theta)^s \left(n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+s} \widetilde{I}_{i+h} \widetilde{I}_{i+s+h} - \left(n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} \right)^2 \right),$$

$$(8.8) \quad \text{cov}^*(\widehat{I}_{1*} \widehat{I}_{(1+h)*}, \widehat{I}_{(1+s)*} \widehat{I}_{(1+s)*+h}) \\ = (1 - \theta)^{\max(s, h)} \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \widehat{I}_{i+s} \widehat{I}_{i+s+h} - \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right) \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right) \right),$$

$$(8.9) \quad \text{cov}^*(\widetilde{I}_{1*} \widetilde{I}_{1*+h}, \widehat{I}_{(1+s)*} \widehat{I}_{(1+s+h)*}) \\ = (1 - \theta)^{s+h} \left(n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} \widehat{I}_{i+s} \widehat{I}_{i+s+h} - \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right) \left(n^{-1} \sum_{i=1}^n \widetilde{I}_i \widetilde{I}_{i+h} \right) \right),$$

$$(8.10) \quad \text{cov}^*(\widehat{I}_{1*} \widehat{I}_{(1+h)*}, \widehat{I}_{(1+s)*} \widehat{I}_{(1+s+h)*}) \\ = \begin{cases} (1 - \theta)^{s+h} \left[n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+s} \widehat{I}_{i+h} \widehat{I}_{i+s+h} - \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+s} \right)^2 \right] + \\ \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+s} (1 - \theta)^s \right)^2 - \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} (1 - \theta)^h \right)^2, & s < h, \\ (1 - \theta)^{s+h} \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+s} \widehat{I}_{i+h} \widehat{I}_{i+s+h} - \left(n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right)^2 \right), & s \geq h. \end{cases}$$

Proof. Relations (8.5) and (8.6) follow from the defining properties of the stationary bootstrap; see Politis and Romano [23].

We will only show that (8.10) holds; (8.7)–(8.9) can be proved in a similar (and even simpler) way but we omit further details. First assume $s < h$. Recall L_1 from the construction of the stationary bootstrap scheme. Consider the following decomposition

$$\begin{aligned} & E^* [\widehat{I}_{1*} \widehat{I}_{(1+h)*} \widehat{I}_{(1+s)*} \widehat{I}_{(1+s+h)*}] \\ &= E^* [\widehat{I}_{1*} \widehat{I}_{(1+h)*} \widehat{I}_{(1+s)*} \widehat{I}_{(1+s+h)*} \mid L_1 \leq s] P(L_1 \leq s) \\ &\quad + E^* [\widehat{I}_{1*} \widehat{I}_{(1+h)*} \widehat{I}_{(1+s)*} \widehat{I}_{(1+s+h)*} \mid s < L_1 \leq h] P(s < L_1 \leq h) \\ &\quad + E^* [\widehat{I}_{1*} \widehat{I}_{(1+h)*} \widehat{I}_{(1+s)*} \widehat{I}_{(1+s+h)*} \mid h < L_1 \leq s + h] P(h < L_1 \leq s + h) \\ &\quad + E^* [\widehat{I}_{1*} \widehat{I}_{(1+h)*} \widehat{I}_{(1+s)*} \widehat{I}_{(1+s+h)*} \mid L_1 > s + h] P(L_1 > s + h) \\ &= Q_1 + Q_2 + Q_3 + Q_4. \end{aligned}$$

We start with Q_1 . For $L_1 \leq s < h$, \widehat{I}_{1*} is independent of $(\widehat{I}_{(1+h)*}, \widehat{I}_{(1+s)*}, \widehat{I}_{(1+s+h)*})$, given (X_t) , but $E^* \widehat{I}_{1*} = 0$ by (8.5) and therefore $Q_1 = 0$. Similarly, for $h < L_1 \leq s + h$, $\widehat{I}_{(1+s+h)*}$ is independent of $(\widehat{I}_{1*}, \widehat{I}_{(1+h)*}, \widehat{I}_{(1+s)*})$, given (X_t) , and since $E^* \widehat{I}_{(1+s+h)*} = 0$, $Q_3 = 0$. Each of the values $i = 1, \dots, n$ has the same chance to be chosen by the bootstrap, i.e., $P^*(\widehat{I}_1^* = \widehat{I}_i) = n^{-1}$ for $i = 1, \dots, n$. Thus, for $L_1 > s + h$ and the chosen i , the natural ordering $(1^*, (1+h)^*, (1+s)^*, (1+s+h)^*) = (i, i+h, i+s, i+s+h)$ is preserved and

²If indices in the sums below exceed the value n they are interpreted in the circular sense, i.e., mod n .

therefore

$$\begin{aligned} Q_4 &= n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+s} \widehat{I}_{i+h} \widehat{I}_{i+s+h} P(L_1 > s+h) \\ &= n^{-1} \sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+s} \widehat{I}_{i+h} \widehat{I}_{i+s+h} (1-\theta)^{s+h}. \end{aligned}$$

By a similar argument, (8.6) and using stationarity, we have

$$\begin{aligned} Q_2 &= E^*[\widehat{I}_1^* \widehat{I}_{(1+s)}^* \mid s < L_1 \leq h] E^*[\widehat{I}_{(1+h)}^* \widehat{I}_{(1+h+s)}^*] P(s < L_1 \leq h) \\ &= n^{-2} \left(\sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+s} \right)^2 (1-\theta)^s \left((1-\theta)^s - (1-\theta)^h \right). \end{aligned}$$

Combining the above expressions and taking into account (8.6), we arrive at (8.10) for $s < h$.

We proceed with the case $s > h$. Then we have the corresponding decomposition

$$\begin{aligned} &E^*[\widehat{I}_1^* \widehat{I}_{(1+h)}^* \widehat{I}_{(1+s)}^* \widehat{I}_{(1+s+h)}^*] \\ &= E^*[\widehat{I}_1^* \widehat{I}_{(1+h)}^* \widehat{I}_{(1+s)}^* \widehat{I}_{(1+s+h)}^* \mid L_1 \leq h] P(L_1 \leq h) \\ &\quad + E^*[\widehat{I}_1^* \widehat{I}_{(1+h)}^* \widehat{I}_{(1+s)}^* \widehat{I}_{(1+s+h)}^* \mid h < L_1 \leq s] P(h < L_1 \leq s) \\ &\quad + E^*[\widehat{I}_1^* \widehat{I}_{(1+h)}^* \widehat{I}_{(1+s)}^* \widehat{I}_{(1+s+h)}^* \mid s < L_1 \leq s+h] P(s < L_1 \leq s+h) \\ &\quad + E^*[\widehat{I}_1^* \widehat{I}_{(1+h)}^* \widehat{I}_{(1+s)}^* \widehat{I}_{(1+s+h)}^* \mid L_1 > s+h] P(L_1 > s+h) \\ &= Q'_1 + Q'_2 + Q'_3 + Q'_4. \end{aligned}$$

We observe that the left-hand side is symmetric in h, s and therefore the same arguments as above show that $Q'_1 = Q'_3 = 0$, $Q_4 = Q'_4$ and

$$\begin{aligned} Q'_2 &= E^*[\widehat{I}_1^* \widehat{I}_{(1+h)}^* \mid h < L_1 \leq s] E^*[\widehat{I}_{(1+s)}^* \widehat{I}_{(1+s+h)}^*] P(h < L_1 \leq s) \\ &= n^{-2} \left(\sum_{i=1}^n \widehat{I}_i \widehat{I}_{i+h} \right)^2 (1-\theta)^h \left((1-\theta)^h - (1-\theta)^s \right) \end{aligned}$$

The case $h = s$ can be considered as a degenerate case, where $Q'_2 = 0$. This completes the proof of (8.10). \square

We conclude with a short discussion of the bias problem of the bootstrapped integrated periodogram mentioned in Remark 22.

Lemma 28. *Assume the conditions of Theorem 21 and the additional condition $\sup_{x \in \Pi} |\psi_h(x)| \leq c/h$ for $h \geq 1$ and a constant c . Then the following relation holds as $n \rightarrow \infty$,*

$$\begin{aligned} &\left(\frac{n}{m} \right)^{0.5} \sup_{\lambda \in \Pi} \left| \psi_0(\lambda) (E^* \widehat{\gamma}_A^*(0) - \widetilde{\gamma}_A(0)) + 2 \sum_{h=1}^{n-1} \psi_h(\lambda) (E^* \widehat{\gamma}_A^*(h) - (1-\theta)^h \widetilde{\gamma}_A(h)) \right| \\ (8.11) \quad &\xrightarrow{P} 0. \end{aligned}$$

Proof. We observe that for $h \geq 0$,

$$\begin{aligned} E^* \widehat{\gamma}_A^*(h) - (1-\theta)^h \widetilde{\gamma}_A(h) &= (1-\theta)^h [(\widehat{\gamma}_A(h) + \widehat{\gamma}_A(n-h)) - \widetilde{\gamma}_A(h)] \\ (8.12) \quad &= (1-\theta)^h [\widetilde{\gamma}_A(n-h) - m(p_0 - \overline{I}_n)^2]. \end{aligned}$$

For fixed h we have $(n/m)^{0.5} m(p_0 - \overline{I}_n)^2 \xrightarrow{P} 0$ as $n \rightarrow \infty$ and

$$(n/m)^{0.5} E|\widetilde{\gamma}_A(n-h)| \leq c(m/n)^{0.5} h p_0 \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore it suffices to show that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left(\sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} \psi_h(\lambda) (E^* \tilde{\gamma}_A^*(h) - (1-\theta)^h \tilde{\gamma}_A(h)) \right| > \delta \right), \quad \delta > 0.$$

Keeping in mind (8.12), we have

$$(n/m)^{0.5} m(p_0 - \bar{I}_n)^2 \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} \psi_h(\lambda) (1-\theta)^h \right| = O_P(1/(\theta\sqrt{mn})) = o_P(1),$$

where we used $\theta^2 n/m \rightarrow \infty$, and

$$\begin{aligned} & (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=k+1}^{n-1} \psi_h(\lambda) (1-\theta)^h \tilde{\gamma}_A(n-h) \right| \\ & \leq (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=1}^{n-k-1} \psi_{n-h}(\lambda) (1-\theta)^{n-h} [\tilde{\gamma}_A(h) - m(1-h/n)(p_h - p_0^2)] \right| \\ & \quad + (n/m)^{0.5} \sup_{\lambda \in \Pi} \left| \sum_{h=1}^{n-k-1} \psi_{n-h}(\lambda) (1-\theta)^{n-h} m(1-h/n)(p_h - p_0^2) \right| \\ & = I_1 + I_2. \end{aligned}$$

Under the assumption $\sup_{x \in \Pi} |\psi_h(x)| \leq c/h$ uniformly for $h \geq 1$, we have for small $\varepsilon > 0$,

$$I_2 \leq (m/n)^{0.5} c \sum_{h=1}^{\infty} \xi_h \rightarrow 0, \quad n \rightarrow \infty.$$

Now we can adapt the proof of Lemma 23 to prove that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} P(I_1 > \delta) = 0, \quad \delta > 0.$$

This proves (8.11). \square

However, under the assumptions of Theorem 15, it is in general not possible to replace the quantities $(1-\theta)^h \tilde{\gamma}_A(h)$ in (8.11) by $\tilde{\gamma}_A(h)$, i.e., in general we do not have the relation $(n/m)^{0.5} (E^* J_{n,A}^* - J_{n,A}) \xrightarrow{P} 0$. Indeed, taking into account (8.11) and assuming η -dependence for (X_t) , we have $E \tilde{\gamma}_A(h) = 0$ for $h > \eta$ and

$$\begin{aligned} & (n/m)^{0.5} (E^* J_{n,A}^* - J_{n,A}) \\ & = 2(n/m)^{0.5} \sum_{h=1}^{n-1} \psi_h(\lambda) [(1-\theta)^h - 1] \tilde{\gamma}_A(h) + o_P(1) \\ & = 2(n/m)^{0.5} \sum_{h=1}^{n-1} \psi_h(\lambda) [(1-\theta)^h - 1] (\tilde{\gamma}_A(h) - E \tilde{\gamma}_A(h)) \\ & \quad + 2(n/m)^{0.5} \sum_{h=1}^{\eta} \psi_h(\lambda) [(1-\theta)^h - 1] (1-h/n) m(p_h - p_0^2) + o_P(1). \end{aligned}$$

An argument similar to the proof of Theorem 15 shows that the first term on the right-hand side is stochastically bounded, while the second term may diverge (for example, if $\gamma_A(\eta) > 0$ and $\psi_\eta \neq 0$) since it is of the order $\theta(n/m)^{0.5}$ which converges to infinity in view of the assumption $\theta^2 n/m \rightarrow \infty$ which is vital for the proof of the consistency of the stationary bootstrap.

REFERENCES

- [1] BILLINGSLEY, P. (1999) *Convergence of Probability Measures*. 2nd Edition. Wiley, New York.
- [2] BREIMAN, L. (1965) On some limit theorems similar to the arc-sin law. *Theory Probab. Appl.* **10**, 323–331.
- [3] BROCKWELL, P. AND DAVIS, R.A. (1991) *Time Series: Theory and Methods*. 2nd Edition. Springer, New York.
- [4] DAHLHAUS, R. (1988) Empirical spectral processes and their applications to time series analysis. *Stoch. Proc. Appl.* **30**, 69–83.
- [5] DAHLHAUS, R. AND POLONIK, W. (2002) Empirical spectral processes and nonparametric maximum likelihood estimation for time series. In: *Dehling, H.G. Mikosch, T. and Sørensen, M. (Eds.) (2002) Empirical Process Techniques for Dependent Data*. Birkhäuser, Boston, pp. 275–298.
- [6] DAVIS, R.A. AND HSING, T. (1995) Point process and partial sum convergence for weakly dependent random variables with infinite variance. *Ann. Prob.* **23**, 879–917.
- [7] DAVIS, R.A. AND MIKOSCH, T. (2009) The extremogram: a correlogram for extreme events. *Bernoulli* **15**, 977–1009.
- [8] DAVIS, R.A. AND MIKOSCH, T. (2009) Extremes of stochastic volatility models. In: Andersen, T.G., Davis, R.A., Kreiss, J.-P. and Mikosch, T. (Eds.) *Handbook of Financial Time Series*. Springer, Heidelberg, pp. 355–364.
- [9] DAVIS, R.A. AND MIKOSCH, T. (2009) Extremes value theory of GARCH processes. In: Andersen, T.G., Davis, R.A., Kreiss, J.-P. and Mikosch, T. (Eds.) *Handbook of Financial Time Series*. Springer, Heidelberg, pp. 187–200.
- [10] DAVIS, R.A., MIKOSCH, T. AND CRIBBEN, I. (2012) Towards estimating extremal serial dependence via the bootstrapped extremogram. *J. Econometrics*. **170**, 142–152.
- [11] DAVIS, R.A., MIKOSCH, T. AND ZHAO, Y. (2013) Measures of serial extremal dependence and their estimation. *Stoch. Proc. Appl.* **123**, 2575–2602.
- [12] DEHLING, H., DURIEU, O. AND VOLNY, D. (2009) New techniques for empirical processes of dependent data. *Stoch. Proc. Appl.* **119**, 3699–3718.
- [13] EMBRECHTS, P., KLÜPPELBERG, C. AND MIKOSCH, T. (1997) *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin.
- [14] GRENANDER, U. AND ROSENBLATT, M. (1984) *Statistical Analysis of Stationary Time Series*. 2nd Edition. Chelsea Publishing Co., New York.
- [15] HANNAN, E.J. (1960) *Time Series Analysis*. Wiley, New York.
- [16] HIDA, T. (1980) *Brownian Motion*. Springer, New York.
- [17] IBRAGIMOV, I.A. AND LINNICK, YU.V. (1971) *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groningen.
- [18] JANSSEN, A. AND DREES, H. (2014) A stochastic volatility model with flexible extremal dependence structure. Available under arXiv:1310.4621v1.
- [19] KLÜPPELBERG, C. AND MIKOSCH, T. (1996) The integrated periodogram for stable processes. *Ann. Stat.*, **24**, 1855–1879.
- [20] KALLENBERG, O. (1983) *Random Measures*, 3rd Edition. Akademie-Verlag, Berlin.
- [21] MIKOSCH, T. AND ZHAO, Y. (2014) A Fourier analysis of extreme events. *Bernoulli* **20**, 803–845.
- [22] PETROV, V.V. (1995) *Limit Theorems of Probability Theory*. Oxford University Press, Oxford (UK).
- [23] POLITIS, D. AND ROMANO, J. (1994) The stationary bootstrap. *J. Amer. Statist. Assoc.*, **89**, 1303–1313.
- [24] PRIESTLEY, M.B. (1981) *Spectral Analysis and Time Series*. Academic Press, London, New York.
- [25] RESNICK, S.I. (1987) *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.
- [26] RESNICK, S.I. (2007) *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*. Springer, New York.
- [27] RIO, E. (1995) About the Lindeberg method for strongly mixing sequences. *ESAIM: Probab. and Statist.* **1**, 35–61.
- [28] SHORACK, G.R. AND WELLNER, J.A. (1986) *Empirical Processes with Applications to Statistics*. Wiley, New York.
- [29] STĂRICĂ, C. (2003) Is GARCH(1,1) as good a model as the Nobel Prize accolades would imply? Technical report. Available under
 $\backslash\protect\vrule\width0pt\protect\href{\http://www.stat-center.unige.ch/files/3913/9585/0846/NobelGarch.pdf}\{\http://www.st.$
- [30] ZYGMUND, A. (2002) *Trigonometric Series. Vol. I, II*. 3rd Edition Cambridge University Press, Cambridge (UK).

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